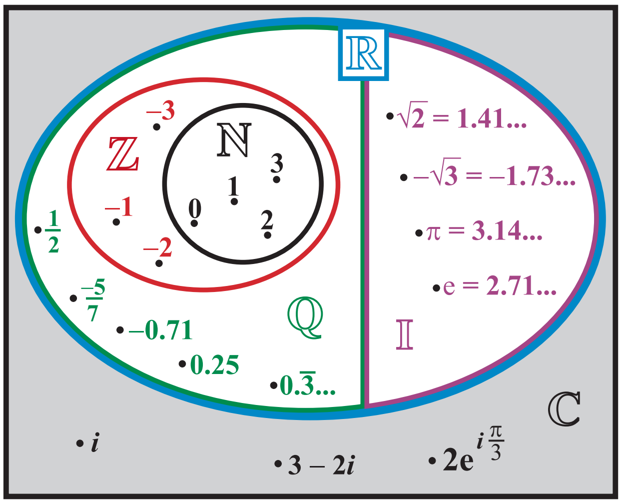
Mathematics formulary

1 Number Systems



- **Natural numbers:** $\mathbb{N} = \{0,\ 1,\ 2,\ 3,\ ...\}$.
- **Integers:** $\mathbb{Z} = \{0,\ \pm 1,\ \pm 2,\ ...\}$.
- **Rational numbers:** Set of all fractions: $\mathbb{Q} = \{\frac{m}{n} \mid m, n \in \mathbb{Z},\ n \neq 0\}$. Numbers with periodic or terminating decimal expansion.
- **Irrational numbers $\mathbb{I}$:** Numbers with infinite nonperiodic decimal expansion.
- **Real numbers $\mathbb{R}$:** Union of rational and irrational numbers.
- **Complex numbers:** $\mathbb{C} = \{x + iy \,|\, x, y \in \mathbb{R}\}$ with $i^2 = -1$.

Complex Numbers

▶ **Imaginary unit:** $i^2 = -1$

▶ **Euler's formula:**

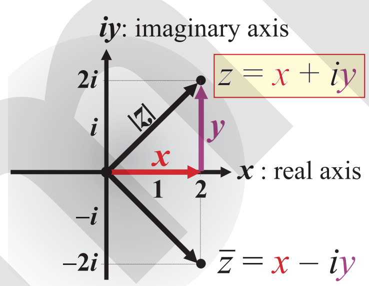
$$e^{i\varphi} = \cos(\varphi) + i \cdot \sin(\varphi)$$

$e^{i\varphi} = \text{cis}(\varphi), \quad |e^{i\varphi}| = 1.$

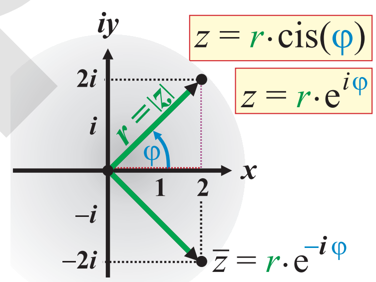
▶ **Argand diagram:**

xy-plane of the complex numbers.

Cartesian coordinates



Polar coordinates

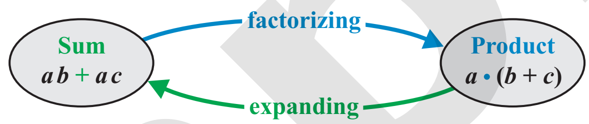


Complex number	z	$z = x + i\,y$ $\begin{cases} x : \text{real part} \\ y : \text{imaginary part} \end{cases}$	$z = r \cdot e^{i\varphi} = r \cdot \text{cis}(\varphi)$
Complex conjugate	$\overline{z}$	$\overline{z} = x - i\,y$	$\overline{z} = r \cdot e^{-i\varphi}$
Modulus	$\lvert z\rvert$	$\lvert z\rvert = \sqrt{z \cdot \overline{z}} = \sqrt{x^2 + y^2}$	$\lvert z\rvert = r = \sqrt{x^2 + y^2}$
Angle	φ	$x = r \cdot \cos(\varphi)$ $y = r \cdot \sin(\varphi)$	$\tan(\varphi) = \frac{y}{x}$ $\varphi = \arg(z)$
Addition **Subtraction**	$z_1 + z_2$ $z_1 - z_2$	$(x_1 \pm x_2) + i\,(y_1 \pm y_2)$	
Multiplication	$z_1 \cdot z_2$	$(x_1\,x_2 - y_1\,y_2) + i\,(x_1\,y_2 + x_2\,y_1)$	$r_1 \cdot r_2 \cdot e^{i\,(\varphi_1 + \varphi_2)}$
Division $(z_2 \neq 0)$	$\frac{z_1}{z_2}$	$\frac{z_1 \cdot \overline{z_2}}{\lvert z_2\rvert^2} = \frac{(x_1\,x_2 + y_1\,y_2) + i\,(x_2\,y_1 - x_1\,y_2)}{{x_2}^2 + {y_2}^2}$	$\frac{r_1}{r_2} \cdot e^{i\,(\varphi_1 - \varphi_2)}$
Inverse $(z \neq 0)$	$\frac{1}{z}$	$\frac{\overline{z}}{\lvert z\rvert^2} = \frac{x - i\,y}{x^2 + y^2}$	$\frac{1}{r} \cdot e^{-i\varphi}$
Powers	z^n	$r^n \cdot (\cos(n \cdot \varphi) + i\,\sin(n \cdot \varphi)) = r^n \cdot e^{i\,n\,\varphi}$	
Roots	$\sqrt[n]{z}$	$\sqrt[n]{r} \cdot \left(\cos\left(\frac{\varphi + 2\pi\,k}{n}\right) + i\,\sin\left(\frac{\varphi + 2\pi\,k}{n}\right)\right), \quad k = 0,\ 1, \ldots, (n-1)$	

2 Algebra

2.1 Addition and Multiplication, Basic Laws

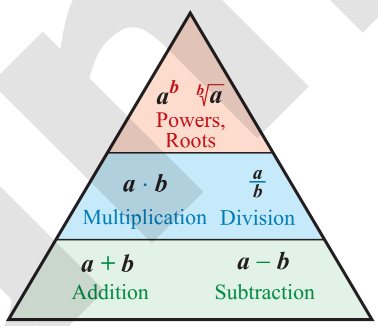
	Addition	Multiplication
Commutative law	$a+b=b+a$	$a\cdot b=b\cdot a$
Associative law	$(a+b)+c=a+(b+c)=a+b+c$	$(a\cdot b)\cdot c=a\cdot(b\cdot c)=a\cdot b\cdot c$
Distributive law	$a\cdot(b\pm c)=a\cdot b\pm a\cdot c$	
Neutral element	$a+0=0+a=a$	$a\cdot 1=1\cdot a=a$
Inverse element	$a+(-a)=(-a)+a=0$	$a\cdot(a^{-1})=(a^{-1})\cdot a=1$



2.2 Order of Operators

Optional brackets:

- $-1^2=-(1)^2=-1$
- $2\cdot 3^4=2\cdot(3^4)=162$
- $4/2+3=(4/2)+3=5$
- $2+3\cdot 4=2+(3\cdot 4)=14$



Mandatory brackets:

- $(-1)^2=(-1)\cdot(-1)=+1$
- $(2\cdot 3)^4=6^4=1296$
- $4/(2+3)=4/5=0.8$
- $(2+3)\cdot 4=5\cdot 4=20$

Mnemonic: The precedence rules can be memorized by the acronym BEDMAS:
Brackets → Exponents, roots → Division, Multiplication → Addition, Subtraction.

2.3 Equivalence Transformations

Equation $a=b$		Inequality $a<b$
$a\pm c=b\pm c$	Addition, Subtraction	$a\pm c<b\pm c$
$a\cdot c=b\cdot c$	Multiplication by $c\neq 0$	$a\cdot c<b\cdot c$ if $c>0$ $a\cdot c>b\cdot c$ if $c<0$ [*]
$\frac{a}{c}=\frac{b}{c}$	Division by $c\neq 0$	$\frac{a}{c}<\frac{b}{c}$ if $c>0$ $\frac{a}{c}>\frac{b}{c}$ if $c<0$ [*]
$\frac{1}{a}=\frac{1}{b}$	Reciprocal $(a,\, b\neq 0)$	$\frac{1}{a}<\frac{1}{b}$ if $a\cdot b<0$ $\frac{1}{a}>\frac{1}{b}$ if $a\cdot b>0$ [*]

[*] : Inequality changes direction.

2.4 Binomial Formulae, Binomial Theorem

Binomial formulae:

1st formula: $(a+b)^2 = a^2 + 2\cdot a\cdot b + b^2$

2nd formula: $(a-b)^2 = a^2 - 2\cdot a\cdot b + b^2$

3rd formula: $(a+b)\cdot(a-b) = a^2 - b^2$

- $a^2 + b^2$ irreducible in $\mathbb{R}$.
- $a^3 + b^3 = (a+b)\cdot(a^2 - a\cdot b + b^2)$
- $a^3 - b^3 = (a-b)\cdot(a^2 + a\cdot b + b^2)$
- $a^n - b^n = (a-b)\cdot\sum_{k=0}^{n-1} a^{n-1-k}\cdot b^k$

Binomial theorem:

$$(a+b)^n = \underbrace{\binom{n}{0}}_{=1} a^n\, b^0 + \binom{n}{1} a^{n-1}\, b^1 + \binom{n}{2} a^{n-2}\, b^2 + \ldots + \underbrace{\binom{n}{n}}_{=1} a^0\, b^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k}\cdot b^k$$

- Binomial coefficients: $\binom{n}{k} = \frac{n!}{k!\cdot(n-k)!}$.
- Factorial: $n! = 1\cdot 2\cdot\ \ldots\ \cdot n$, $\quad 0! = 1! = 1.$ (⇒ See combinatorics on p. 36)
- For $(a-b)^n$ the sign is *alternating*: $(a-b)^3 = +a^3 - 3\,a^2\,b + 3\,a\,b^2 - b^3$.

Pascal's triangle and binomial theorem:

1	$n=0$	$\binom{0}{0}$	$(a+b)^0 = 1$
1 1	$n=1$	$\binom{1}{0}$ $\binom{1}{1}$	$(a+b)^1 = 1a^1 + 1b^1$
1 2 1	$n=2$	$\binom{2}{0}$ $\binom{2}{1}$ $\binom{2}{2}$	$(a+b)^2 = 1a^2 + 2a^1b^1 + 1b^2$
1 3 3 1	$n=3$	$\binom{3}{0}$ $\binom{3}{1}$ $\binom{3}{2}$ $\binom{3}{3}$	$(a+b)^3 = 1a^3 + 3a^2b^1 + 3a^1b^2 + 1b^3$
1 4 6 4 1	$n=4$	$\binom{4}{0}$ $\binom{4}{1}$ $\binom{4}{2}$ $\binom{4}{3}$ $\binom{4}{4}$	$(a+b)^4 = 1a^4 + 4a^3b^1 + 6a^2b^2 + 4a^1b^3 + 1b^4$

Absolute value: $|x| = \sqrt{x^2} = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$ „makes x positive". See p. 15.

2.5 Fractions

Addition, Subtraction	$\frac{a}{b} \pm \frac{x}{y} = \frac{a\cdot y}{b\cdot y} \pm \frac{x\cdot b}{y\cdot b} = \frac{a\cdot y \pm x\cdot b}{b\cdot y}$ $\quad b,\ y \neq 0$	▶ Put onto the common denominator, then add the numerators.
Multiplication	$\frac{a}{b}\cdot\frac{x}{y} = \frac{a\cdot x}{b\cdot y}$ $\quad b,\ y \neq 0$	▶ Multiply numerators and denominators separately.
Division, compound fractions	$\frac{a}{b} : \frac{x}{y} = \frac{\frac{a}{b}}{\frac{x}{y}} = \frac{a}{b}\cdot\frac{y}{x}$ $\quad b,\ x,\ y \neq 0$	▶ Dividing by a fraction = multiplying by its reciprocal.

2.6 Powers

Definition: $a^n = \underbrace{a \cdot a \cdot \ldots \cdot a}_{n \text{ factors}}$ is called the n^{th} power of a, where $\begin{cases} a \in \mathbb{R} & : \text{Base} \\ n \in \mathbb{N} & : \text{Exponent.} \end{cases}$

Particularly: $a^1 = a$ and $\begin{cases} a^0 = 1, & \text{if } a \neq 0 \\ 0^n = 0, & \text{if } n > 0. \end{cases}$

- **Negative exponents:** $k \cdot a^{-n} = \frac{k}{a^n}$ $a \neq 0.$
- **Rational exponents:** $a^{\frac{m}{n}} = \sqrt[n]{a^m}$ $a \geq 0, \quad n > 0.$

Particularly: $a^{\frac{1}{n}} = \sqrt[n]{a}$ Square root ($n = 2$): $\sqrt{a} = a^{\frac{1}{2}}$

Power laws

Same base	$a^n \cdot a^m = a^{n+m}$	$\frac{a^n}{a^m} = a^{n-m}$	$a \neq 0$
Same exponent	$a^n \cdot b^n = (a \cdot b)^n$	$\frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n$	$b \neq 0$
	$\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{a \cdot b}$	$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$	$b \neq 0$
Powers of powers	$(a^n)^m = a^{n \cdot m} = (a^m)^n$	$\sqrt[m]{\sqrt[n]{a}} = \sqrt[nm]{a} = \sqrt[n]{\sqrt[m]{a}}$	$a \geq 0$

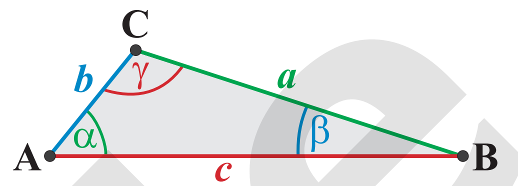
2.7 Logarithms (see p. 18)

Definition	$\log_a(x) = y \quad \Leftrightarrow \quad a^y = x$	$a, x > 0, \quad a \neq 1$
Multiplication, Division	$\log_a(x \cdot y) = \log_a(x) + \log_a(y)$	$\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$
Powers	$\log_a(x^y) = y \cdot \log_a(x) \quad x > 0$	$a^x = b \quad \Rightarrow \quad x = \frac{\ln(b)}{\ln(a)}$
Change of base	$\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$ $\quad a > 0;\ a \neq 1$, $b > 0;\ b \neq 1$	especially: $\log_a(x) = \frac{\ln(x)}{\ln(a)}$

⇒ See Logarithmic functions on p. 18.

3 Plane Geometry

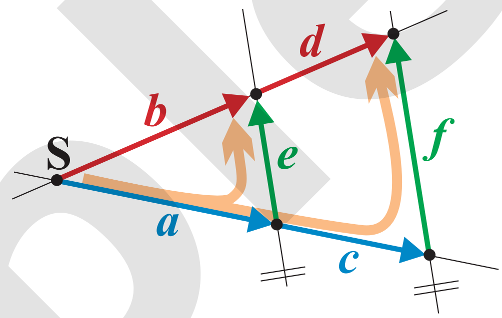
3.1 Triangles



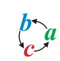
- **Sum of angles:** $\alpha + \beta + \gamma = 180°$
- **Triangle inequality:** $c < a + b$
- **Intercept theorems (proportionality):**
  Two triangles are called similar if they have the **same angles**. Equivalently, the **ratios** of their sides are **equal**.

  1st **Intercept theorem:** $\frac{a}{b} = \frac{c}{d} = \frac{a+c}{b+d}$

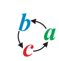
  2nd **Intercept theorem:** $\frac{a}{e} = \frac{a+c}{f}$



- **Sine rule:** $\frac{a}{\sin(\alpha)} = \frac{b}{\sin(\beta)} = \frac{c}{\sin(\gamma)} = 2 \cdot R$ $\quad R:$ Radius of the circumcircle.
- **Cosine rule:** $c^2 = a^2 + b^2 - 2 \cdot a \cdot b \cdot \cos(\gamma)$ $\quad$ cyclic permutations:

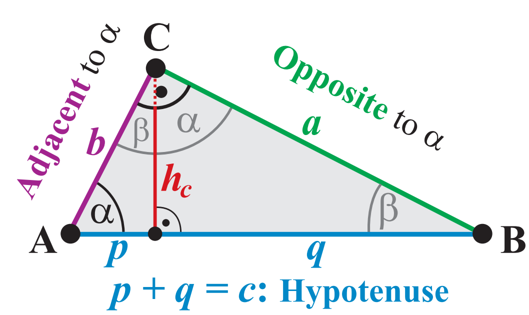


- **Area**: $A_\Delta = \frac{1}{2}(\text{base} \cdot \text{height}) = \frac{c \cdot h_c}{2} = \frac{b \cdot h_b}{2} = \frac{a \cdot h_a}{2}$
  - ▶ **two sides and their enclosed angle:** $A_\Delta = \frac{b \cdot c}{2} \cdot \sin(\alpha)$ $\quad$ cyclic perm.:



  - ▶ **three sides (Heron):** $A_\Delta = \sqrt{s(s-a)(s-b)(s-c)}$ with the semi-perimeter $s = \frac{1}{2}(a+b+c)$.
  - ▶ **three angles and R :** $A_\Delta = 2\,R^2 \cdot \sin(\alpha) \cdot \sin(\beta) \cdot \sin(\gamma)$ $\quad R:$ Radius of the circumcircle.

3.2 Right-angled Triangle



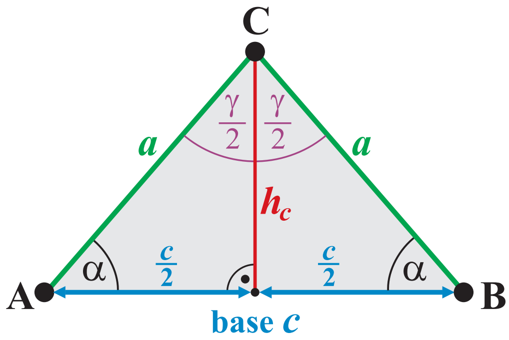
- **Pythagoras' theorem:** $c^2 = a^2 + b^2$
- **Altitude theorem:** ${h_c}^2 = p \cdot q$
- **Euclid's theorem:** $\begin{cases} a^2 = c \cdot q \\ b^2 = c \cdot p \end{cases}$
- **Trigonometric functions:** (see p. 19)

$$\sin(\alpha) = \frac{a}{c} \qquad \cos(\alpha) = \frac{b}{c} \qquad \tan(\alpha) = \frac{a}{b}$$

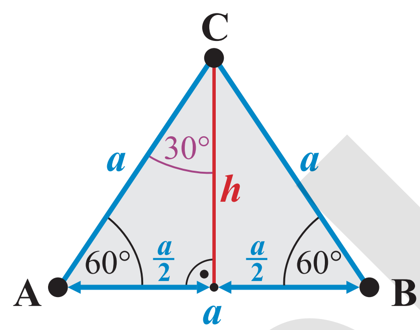
mnemonic:
SOH CAH TOA

3.3 Isosceles and Equilateral Triangles

Isosceles triangle



Equilateral triangle

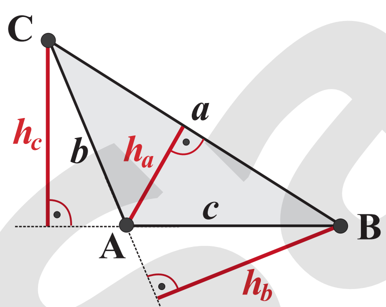


- h_c bisects base c.
- h_c bisects angle γ.
- Equal base angles ($\alpha = \beta$).

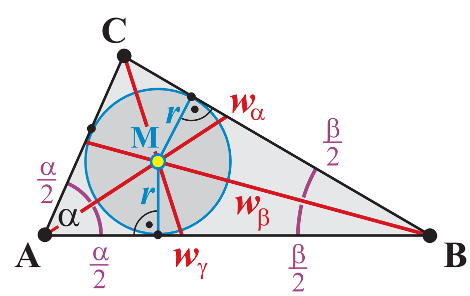
- Height: $h = \frac{\sqrt{3}}{2}\, a$.
- Area: $A = \frac{\sqrt{3}}{4}\, a^2$.
- Radius of the circumcircle: $R = \frac{\sqrt{3}}{3}\, a = \frac{2}{3}\, h$.
- Inradius: $r = \frac{\sqrt{3}}{6}\, a = \frac{1}{3}\, h$.

3.4 Lines in a Triangle

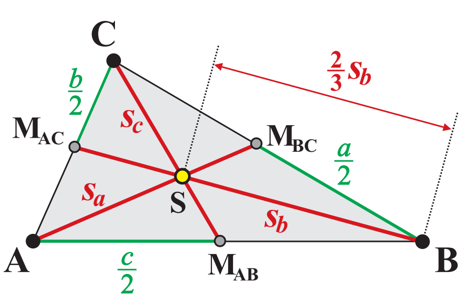
Heights are straight lines through a vertex perpendicular to the opposite side.



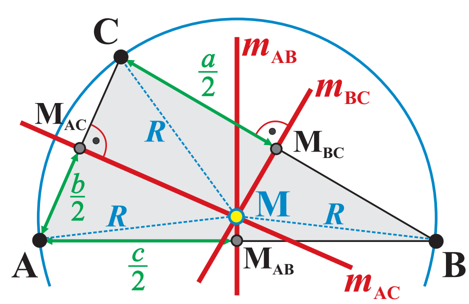
Angle bisectors bisect an angle of the triangle. Each point on an angle bisector has the **same distance** from the adjacent sides. Angle bisectors intersect at the center M of the incircle.



Medians are the lines from a vertex to the midpoint of the opposite side. They intersect in the **ratio 2:1**. The point of coincidence is the **centroid S** (center-of-mass) of the triangle. See p. 32.

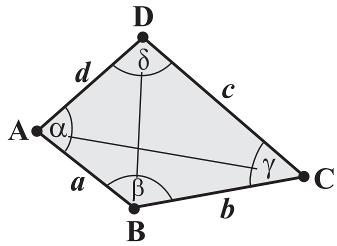


Perpendicular bisectors are the set of points having the **same distance** from two vertices of the triangle. They intersect at the center M of the circumcircle.



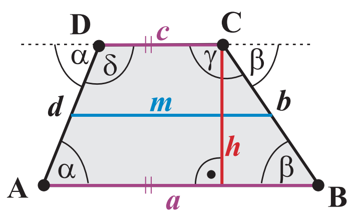
3.5 Quadrilaterals

Quadrilateral



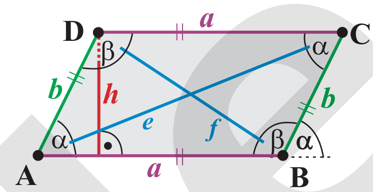
- $\alpha + \beta + \gamma + \delta = 360°$

Trapezium, trapezoid



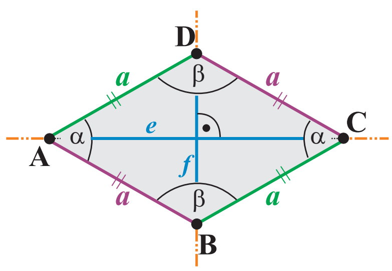
- $A = \frac{a+c}{2} \cdot h = m \cdot h$

Parallelogram, rhomboid



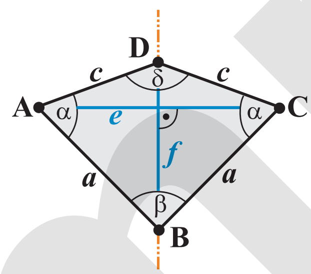
- $A = a \cdot h = a \cdot b \cdot \sin(\alpha)$

Rhombus



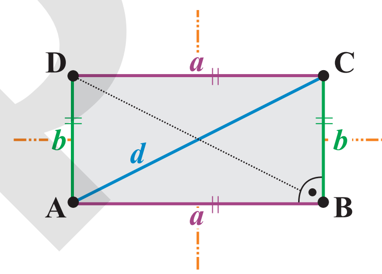
- $A = \frac{e \cdot f}{2} = a^2 \cdot \sin(\alpha)$

Kite



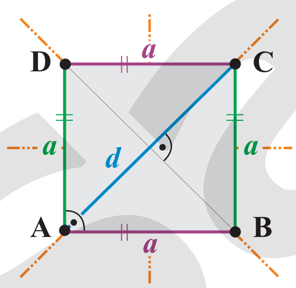
- $A = \frac{e \cdot f}{2} = a \cdot c \cdot \sin(\alpha)$

Rectangle



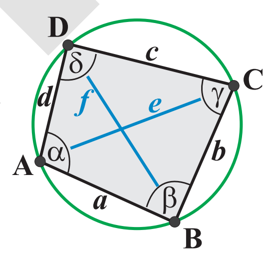
- $A = a \cdot b$

Square



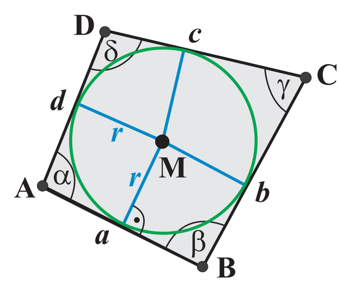
- $A = a^2$
- $d = a \cdot \sqrt{2}$

Cyclic quadrilateral



- $\alpha + \gamma = \beta + \delta = 180°$
- $a \cdot c + b \cdot d = e \cdot f$

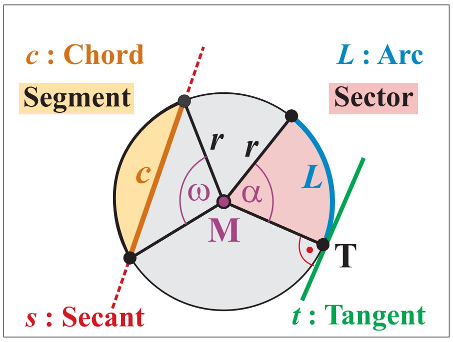
Tangent quadrilateral

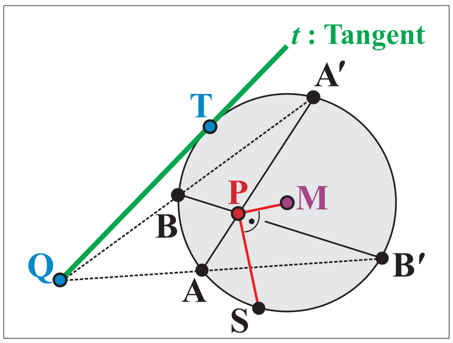


- $a + c = b + d$
- $A = r \cdot \frac{a+b+c+d}{2}$

Symmetry axis are shown in orange color.

3.6 Circle





Circumference	$C = 2\pi \cdot r$
Arc length	$L = 2\pi\, r \cdot \frac{\alpha}{360°}$
Area	$A = \pi \cdot r^2$
Sector	$A_{\text{sector}} = \pi\, r^2 \cdot \frac{\alpha}{360°} = \frac{b \cdot r}{2}$
Segment	$A_{\text{segment}} = r^2 \cdot \left(\pi \cdot \frac{\omega}{360°} - \frac{1}{2} \cdot \sin(\omega)\right)$
Intersecting chord theorem	$\overline{PA} \cdot \overline{PA'} = \overline{PB} \cdot \overline{PB'} = \overline{PS}^2$
Intersecting secant theorem	$\overline{QB} \cdot \overline{QA'} = \overline{QA} \cdot \overline{QB'} = \overline{QT}^2$

▶ **Equation of circle** c with center $M(u \,/\, v)$ and radius r:

Center form — expanding → **Expanded form**

$c\colon\ (x-u)^2 + (y-v)^2 = r^2$

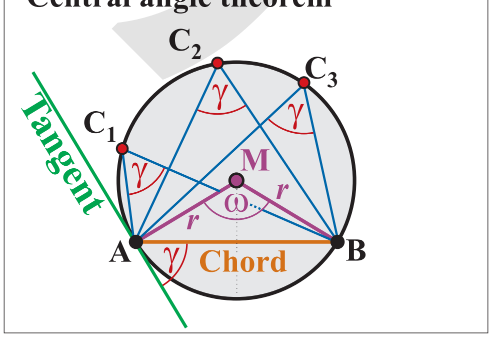
$c\colon\ x^2 + y^2 + ax + by + c = 0$

← quadratic completion

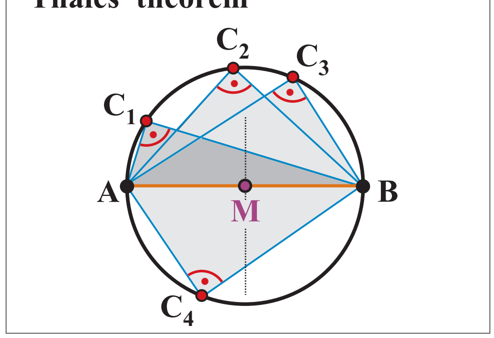
▶ Tangent t to c at point $T(x_0 \,/\, y_0)$: $t\colon\ (x-u) \cdot (x_0-u) + (y-v) \cdot (y_0-v) = r^2$

Circle Angle Theorems

Central angle theorem



Thales' theorem

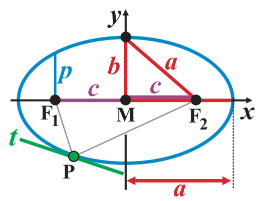
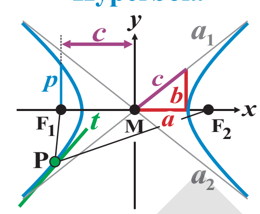
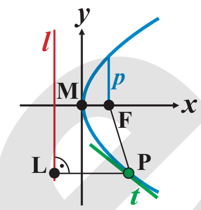


▶ Equal inscribed angle γ.

▶ Central angle $\omega = 2 \cdot \gamma$.

▶ Equal inscribed angle $\gamma = 90°$.

3.7 Conic Sections

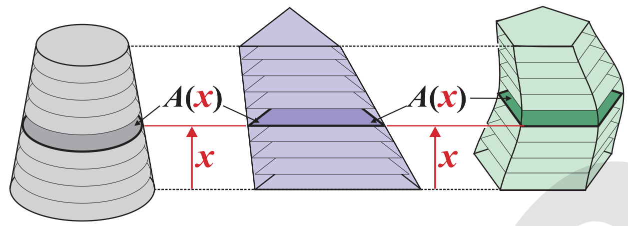
	Ellipse	Hyperbola	Parabola
			
Distance property	$\overline{\mathrm{PF}_1} + \overline{\mathrm{PF}_2} = 2a$	$\lvert\overline{\mathrm{PF}_1} - \overline{\mathrm{PF}_2}\rvert = 2a$	$\overline{\mathrm{PF}} = \overline{\mathrm{PL}}$
Equation for center M(0 / 0)	$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$	$\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$	$\boxed{y^2 = 2 \cdot p \cdot x}$
Parametric form for center M(0 / 0)	$x(\varphi) = a \cdot \cos(\varphi)$ $y(\varphi) = b \cdot \sin(\varphi)$	$x(\varphi) = \pm a \cdot \cosh(\varphi)$ $y(\varphi) = b \cdot \sinh(\varphi)$	
Tangent equation in $P(x_0\ /\ y_0)$	$t:\ \frac{x\,x_0}{a^2} + \frac{y\,y_0}{b^2} = 1$	$t:\ \frac{x\,x_0}{a^2} - \frac{y\,y_0}{b^2} = 1$	$t:\ y\,y_0 = p\,(x + x_0)$
Tangent condition for $t:\ y = m_t\,x + q$	$q^2 = a^2\,m_t^2 + b^2$	$q^2 = a^2\,m_t^2 - b^2$	$q = \frac{p}{2\,m_t}$
Conjugated direction	$m_1 \cdot m_2 = -\frac{b^2}{a^2}$	$m_1 \cdot m_2 = +\frac{b^2}{a^2}$	
Linear eccentrity	$c^2 = a^2 - b^2$	$c^2 = a^2 + b^2$	
Numerical eccentrity	$\varepsilon = \frac{c}{a} < 1$	$\varepsilon = \frac{c}{a} > 1$	$\varepsilon = 1$
Focus	$\mathrm{F}_{1,2}(\pm c\ /\ 0)$	$\mathrm{F}_{1,2}(\pm c\ /\ 0)$	$\mathrm{F}(\frac{p}{2}\ /\ 0)$
Radius of curvature	$r_a = \frac{b^2}{a},\quad r_b = \frac{a^2}{b}$	$r = \frac{b^2}{a}$	$r = p$
Parameter p	$p = \frac{b^2}{a}$	$p = \frac{b^2}{a}$	p
Area	$A = \pi \cdot a \cdot b$		
Asymptotes		$a_{1,2}:\ y = \pm\frac{b}{a} \cdot x$	

Translation from M(0 / 0) to M'(u / v): $\begin{cases} x \to (x - u) \\ y \to (y - v) \end{cases}$

4 Stereometry

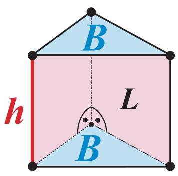
Cavalieri's Principle

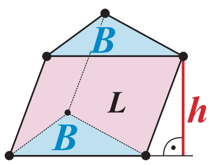
Two solids have the same volume if their cross-sections $A(x)$ have the same area at all levels x.



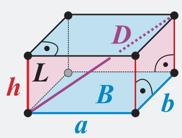
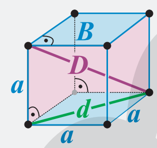
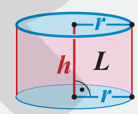
4.1 Prisms and Cylinders (Congruent, Parallel Base and Top Face B)

Right prism | Oblique prism



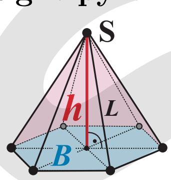


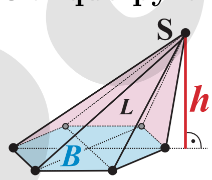
- B : Base area; L : Lateral area.
- h : Height.
- Volume: $\boxed{V = B \cdot h}$
- Surface area: $\boxed{A = 2 \cdot B + L}$

Cuboid	Cube	Cylinder
		
▶ $V = a \cdot b \cdot h$	▶ $V = a^3$	▶ $V = \pi\, r^2 \cdot h$
▶ $A = 2(a \cdot b + a \cdot h + b \cdot h)$	▶ $A = 6 \cdot a^2$	▶ $A = 2 \cdot \pi\, r^2 + 2\pi\, r \cdot h$
▶ $D = \sqrt{a^2 + b^2 + h^2}$	▶ $D = a \cdot \sqrt{3}, \quad d = a \cdot \sqrt{2}$	▶ $L = 2\pi\, r \cdot h$

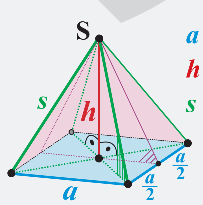
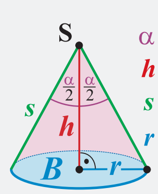
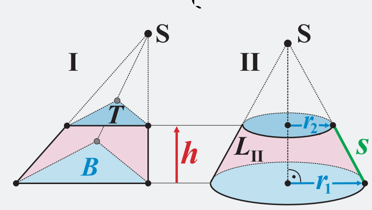
4.2 Pyramids and Cones

Right pyramid | Oblique pyramid





- B : Base area; L : Lateral area.
- h : Height.
- Volume: $\boxed{V = \frac{1}{3} \cdot B \cdot h}$
- Surface area: $\boxed{A = B + L}$

Right, square pyramid	Right circular cone	Frustum of $\left\{ \begin{array}{l} \text{pyramid,} \\ \text{cone} \end{array} \right.$
 a : Side length h : Height s : Slant edge	 α : Aperture angle h : Height s : Slant edge r : Radius	
▶ $V = \frac{1}{3}\, a^2 \cdot h$	▶ $V = \frac{1}{3}\, \pi r^2 \cdot h$	▶ $V_{\mathrm{I}} = \frac{h}{3}\, (B + \sqrt{BT} + T)$
▶ $A = a^2 + L$	▶ $A = \pi r^2 + \pi r s, \quad L = \pi r s$	▶ $V_{\mathrm{II}} = \frac{\pi h}{3}\, ({r_1}^2 + r_1\, r_2 + {r_2}^2)$
▶ $s = \sqrt{h^2 + \frac{a^2}{2}}$	▶ $s = \sqrt{h^2 + r^2}$	▶ $L_{\mathrm{II}} = \pi \cdot s \cdot (r_1 + r_2)$

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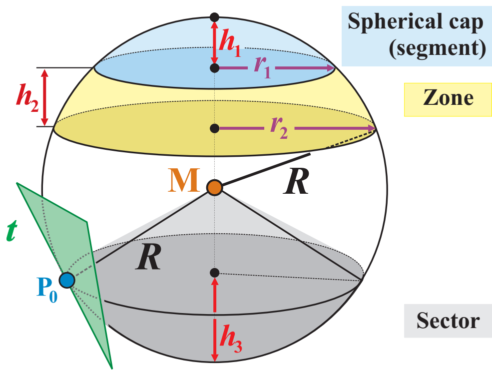
4.3 Sphere



Volume: $V = \frac{4}{3}\,\pi \cdot R^3$

- **Cap:** $V = \frac{1}{3}\,\pi \cdot {h_1}^2 \cdot (3\,R - h_1)$
- **Zone:** $V = \frac{1}{6}\,\pi \cdot h_2 \cdot (3\,{r_1}^2 + 3\,{r_2}^2 + {h_2}^2)$
- **Sector:** $V = \frac{2}{3}\,\pi R^2 \cdot h_3$

Surface area: $A = 4\pi \cdot R^2$

- **Cap:** $L = 2\,\pi\,R \cdot h_1$ (lateral area)
- **Zone:** $L = 2\,\pi\,R \cdot h_2$ (lateral area)
- **Sector:** $A = 2\,\pi\,R \cdot h_3 + \pi R\sqrt{2R\,h_3 - {h_3}^2}$

▶ **Equation of a sphere s** with center M(u / v / w) and radius R:

Center form		**Expanded form**
$s\colon\ (x-u)^2 + (y-v)^2 + (z-w)^2 = R^2$	expanding → / ← quadratic completion	$s\colon\ x^2 + y^2 + z^2 + ax + by + cz + d = 0$

▶ **Tangent plane t** to a sphere s at a point $P_0(x_0\ /\ y_0\ /\ z_0)$:

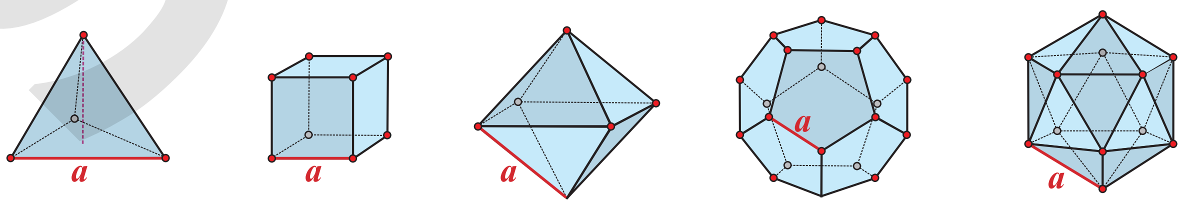
$$t:\ (x-u)\cdot(x_0-u) + (y-v)\cdot(y_0-v) + (z-w)\cdot(z_0-w) = R^2$$

(see planes on p. 35)

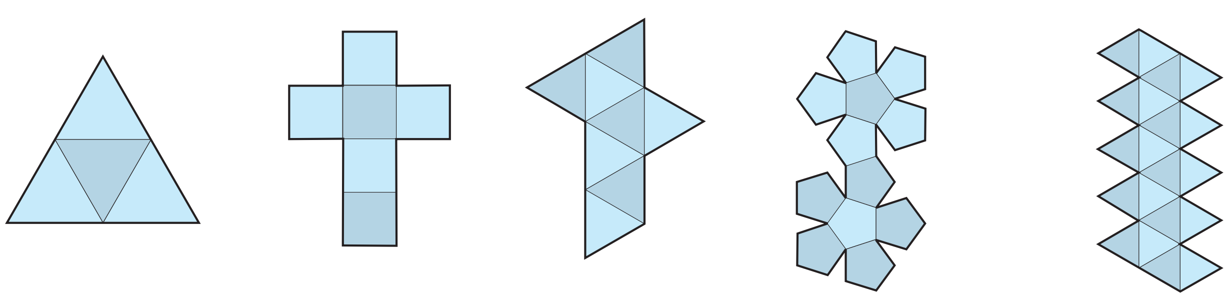
4.4 Polyhedra, Platonic Solids

Euler's polyhedron theorem: $V + F = E + 2$ with
- V : number of vertices,
- F : number of faces,
- E : number of edges.

The Platonic Solids: 5 regular convex solids (all equal edges and equal angles)

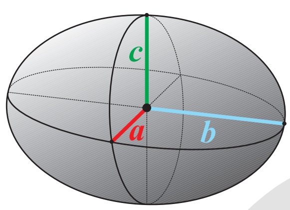
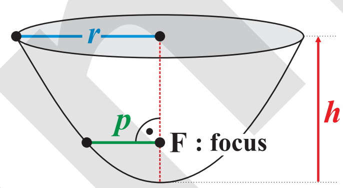
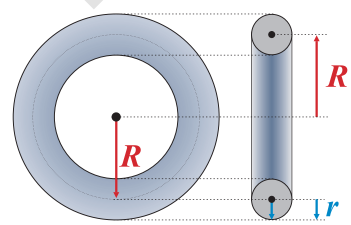


***Tetra*hedron (4 faces)** | ***Hexa*hedron (6 faces)** | ***Octa*hedron (8 faces)** | ***Dodeca*hedron (12 faces)** | ***Icosa*hedron (20 faces)**

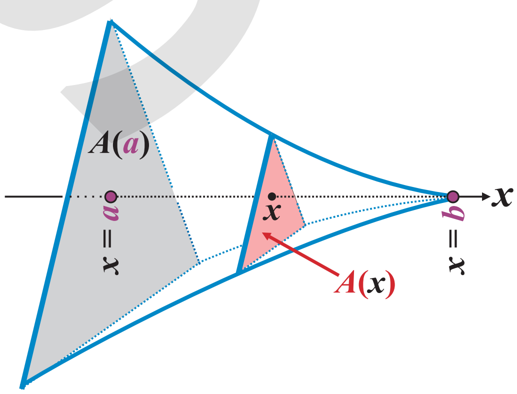


	Volume V	Surface area A	Radius R of the circumsphere	Radius r of the insphere
Tetra-hedron	$\frac{\sqrt{2}}{12}\,a^3$	$\sqrt{3}\,a^2$	$\frac{\sqrt{6}}{4}\,a$	$\frac{\sqrt{6}}{12}\,a$
Hexa-hedron	a^3	$6\,a^2$	$\frac{\sqrt{3}}{2}\,a$	$\frac{1}{2}\,a$
Octa-hedron	$\frac{\sqrt{2}}{3}\,a^3$	$2\sqrt{3}\,a^2$	$\frac{\sqrt{2}}{2}\,a$	$\frac{\sqrt{6}}{6}\,a$
Dodeca-hedron	$\frac{15+7\sqrt{5}}{4}\,a^3$	$3\sqrt{5(5+2\sqrt{5})}\;a^2$	$\frac{(1+\sqrt{5})\sqrt{3}}{4}\,a$	$\frac{\sqrt{10+4.4\sqrt{5}}}{4}\,a$
Icosa-hedron	$\frac{5(3+\sqrt{5})}{12}\,a^3$	$5\sqrt{3}\,a^2$	$\frac{\sqrt{2(5+\sqrt{5})}}{4}\,a$	$\frac{(3+\sqrt{5})\sqrt{3}}{12}\,a$

4.5 Solids with Curved Surface

Ellipsoid	Paraboloid	Torus
		
▸ $V = \frac{4}{3}\pi \cdot a \cdot b \cdot c$	▸ $V = \frac{1}{2}\pi \cdot r^2 \cdot h = \pi \cdot p \cdot h^2$	▸ $V = 2\pi^2 \cdot r^2 \cdot R$ ▸ $A = 4\pi^2 \cdot r \cdot R$

4.6 Volume of a Solid using Integral Calculus



- $V = \int_a^b A(x)\,dx$

  Cross-section area $A(x)$ ⊥ x-Axis.

- Solids of revolution: Volume of a solid obtained by the graph of a function $f(x)$ rotating about the x-axis:

  $V_x = \pi \cdot \int_a^b (\,f(x)\,)^2\,dx$ (see p. 31)

Sphere
Polyhedra

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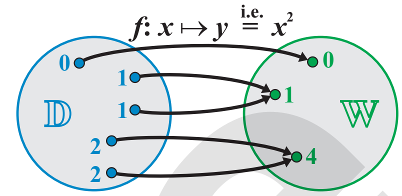
5 Functions

Definition: A **function** $f: \mathbb{D} \to \mathbb{W}$ is a **mapping** from one set $\mathbb{D}$ (**domain**) to another set $\mathbb{W}$ (**range**) so that **each** element $x \in \mathbb{D}$ is assigned **a unique** element $y \in \mathbb{W}$:

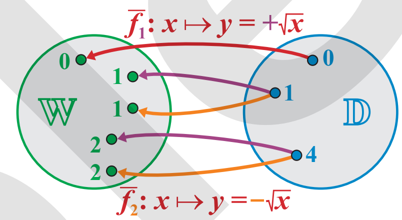
$$f: x \mapsto y = f(x)$$



Inverse function: $\overline{f}: \mathbb{W} \to \mathbb{D}$ reverses the function f:

$\overline{f}(\,f(x)\,) = x$ and $f(\,\overline{f}(y)\,) = y$

Only one-to-one mappings have inverse functions. In order to make a function f invertible, its domain has to be restricted such that f becomes monotonic.



Finding the inverse function:

- *Graphically:* Reflect the graph in the first angle bisector $y = x$.
- *Algebraically:* Solve $y = f(x)$ for x. Then, interchange x and y.

Domain: Set of all allowed x-values:

- $\frac{U(x)}{V(x)} \quad \Rightarrow \quad V(x) \neq 0$
- $\sqrt{g(x)} \quad \Rightarrow \quad g(x) \geq 0$
- $\log_a(g(x)) \quad \Rightarrow \quad g(x) > 0$

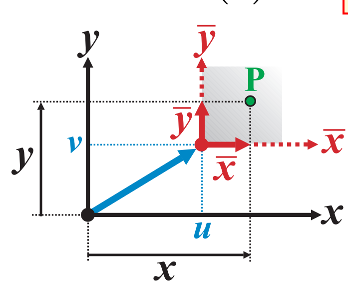
Table of functions and their inverse functions:

Function	$y = f(x)$	$\mathbb{D}_f$	$\mathbb{W}_f$	$y = \overline{f}(x)$
Reciprocal	$\frac{1}{x} = x^{-1}$	$\mathbb{R}\backslash\{0\}$	$\mathbb{R}\backslash\{0\}$	$\frac{1}{x} = x^{-1}$
Square	x^2	$\mathbb{R}$	$y \geq 0$	$\sqrt{x}$
Power	x^n	$\mathbb{R}$	if n even: $y \geq 0$ if n odd: $\mathbb{R}$	$\sqrt[n]{x}$
Sine	$\sin(x)$	$\mathbb{R}$	$[-1,\ 1]$	$\arcsin(x)$
Cosine	$\cos(x)$	$\mathbb{R}$	$[-1,\ 1]$	$\arccos(x)$
Tangent	$\tan(x)$	$\mathbb{R}\backslash\{(n+\frac{1}{2})\,\pi,\ n \in \mathbb{Z}\}$	$\mathbb{R}$	$\arctan(x)$
Exponential	a^x	$\mathbb{R}$	$y > 0$	$\log_a(x)$

5.1 Translation, Rotation of the Coordinate System

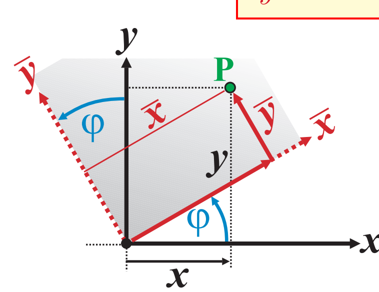
Translation of $\begin{pmatrix} u \\ v \end{pmatrix}$:

$$\overline{x} = x - u \qquad \overline{y} = y - v$$

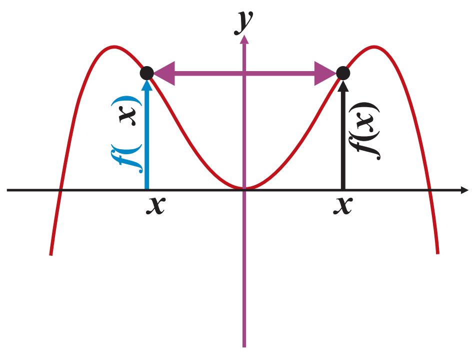
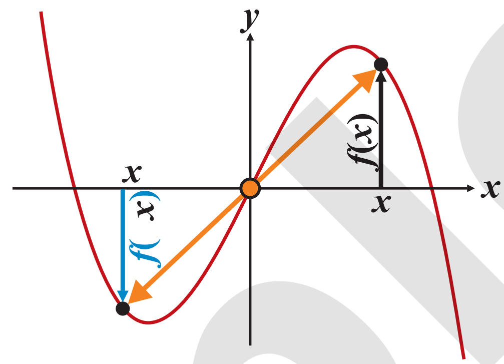


Rotation by φ:

$$\overline{x} = x\cos(\varphi) + y\sin(\varphi) \qquad \overline{y} = -x\sin(\varphi) + y\cos(\varphi)$$



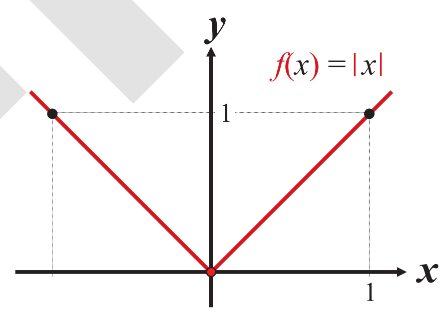
5.2 Symmetry

Even functions:	Odd functions:
Graph symmetrical about the y-axis.	Graph symmetrical about the origin O(0 / 0).
	
$f(-x) = f(x)$	$f(-x) = -f(x)$

5.3 Absolute Value

$$|\,x\,| = \sqrt{x^2} = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

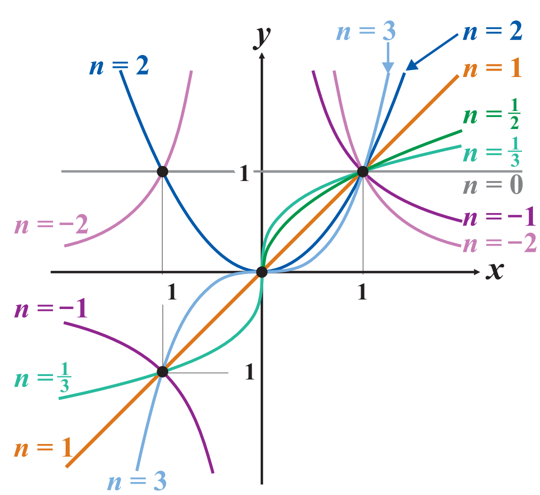
„makes x positive".



- $|\,x\,|$ is continuous but not differentiable at $x = 0$.
- $|\,a \cdot b\,| = |\,a\,| \cdot |\,b\,|$ $\qquad \left|\frac{a}{b}\right| = \frac{|a|}{|b|}$
- $|\,|a| - |b|\,| \leq |\,a + b\,| \leq |\,a\,| + |\,b\,|$.

5.4 Power Functions

Power function: $f(x) = x^n$ $\quad n \in \mathbb{Q}$.



$n = 0$	Constant function.
$0 < n < 1$	Root functions.
$n = 1$	Linear function.
$n \in \mathbb{N};\ n > 1$	Parabolas of n^{th} order.
$n \in \mathbb{Z};\ n < 0$	Hyperbolas of n^{th} order.

The graph of $f(x) = x^n$ is...

- ...symmetrical about the y-axis if n is even,
- ...symmetrical about the origin if n is odd.

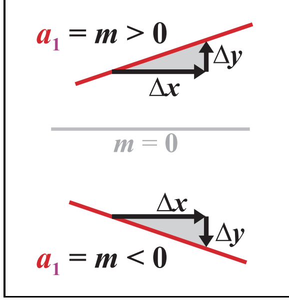
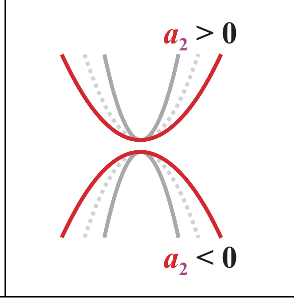
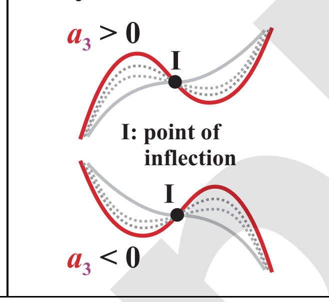
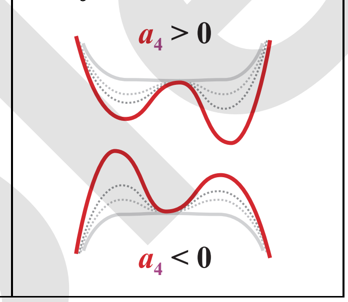
⇒ Derivatives and antiderivatives see p. 29.

5.5 Polynomial Functions (Parabolas of Degree n)

$$y = f(x) = a_n x^n + a_{n-1}x^{n-1} + \ \dots\ + a_1 x + a_0 = \sum_{k=0}^{n} a_k\, x^k \quad \text{with} \begin{cases} n: & \text{Degree, order,} \\ a_n \neq 0. \end{cases}$$

$a_n,\ a_{n-1},\ a_{n-2},\ \dots, a_1, a_0$ are the **coefficients** of $f(x)$.

Overview of the graphs of polynomial functions:

Lines: $n = 1$	Parabolas: $n = 2$	Polynomials: $n = 3$	Polynomials: $n = 4$
			

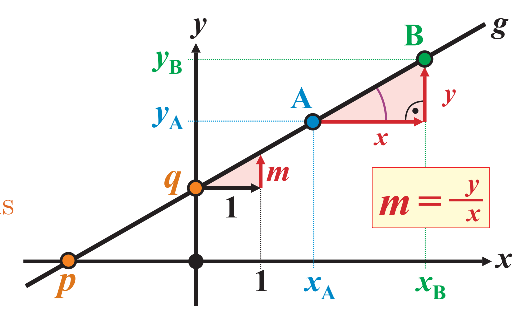
5.5.1 Linear Functions ($n = 1$): $y = m \cdot x^1 + q$ see p. 34.

▶ **Normal form:** $g:\ y = m \cdot x + q$

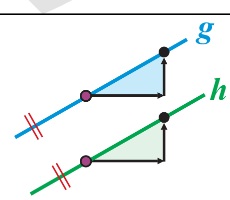
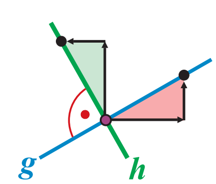
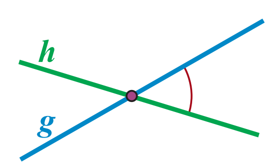
▶ **Point-slope form:** $g:\ y = m \cdot (x - x_A) + y_A$ with $A(x_A \,/\, y_A) \in g$.

- **Slope:** $m = \dfrac{\Delta y}{\Delta x} = \dfrac{y_B - y_A}{x_B - x_A} = \tan(\alpha)$
- ***y*-Intercept:** q.

▶ **Intercept form:** $g: \dfrac{x}{p} + \dfrac{y}{q} = 1$ with the axis intercepts $p,\ q\ \in\ \mathbb{R}\backslash\{0\}\ \cup\ \{\pm\infty\}$.



▶ **Parallel lines, perpendicular lines, angle of intersection:**

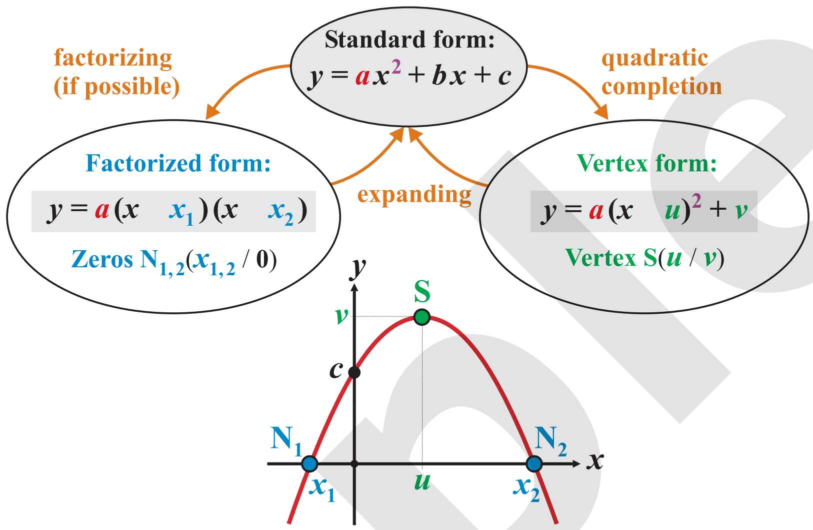
Parallel lines	Perpendicular lines	Angle of intersection $\varphi = \angle(g,\ h)$
		
$g \parallel h \ \Leftrightarrow m_g = m_h$	$g \perp h \ \Leftrightarrow m_h = \dfrac{-1}{m_g}$	$\tan(\varphi) = \left\lvert \dfrac{m_h - m_g}{1 + m_h \cdot m_g} \right\rvert$

▶ **Vector equation and cartesian form on p. 34.**

5.5.2 Quadratic Functions (Parabolas, $n = 2$): $y = a \cdot x^2 + b \cdot x + c$

- $a < 0$: Parabola opens downwards ($\cap$),
  $a > 0$: Parabola opens upwards ($\cup$),
  $a = 1$: Norm parabola.
- $b\,x$: Linear term.
- c : y-Intercept.
- Vertex $S(u \,/\, v)$: $S(-\frac{b}{2a} \,/\, \frac{-b^2+4ac}{4a})$.

$\Rightarrow$ Formula for quadratic equations on p. 21.



5.6 Rational Functions

A **rational function** $f(x)$ is the quotient of two polynomials:

$$f(x) = \frac{U(x)}{V(x)} = \frac{\text{numerator polynomial}}{\text{denominator polynomial}} = \frac{a_n\,x^n + a_{n-1}\,x^{n-1} + \ldots + a_1\,x + a_0}{b_m\,x^m + b_{m-1}\,x^{m-1} + \ldots + b_1\,x + b_0}$$

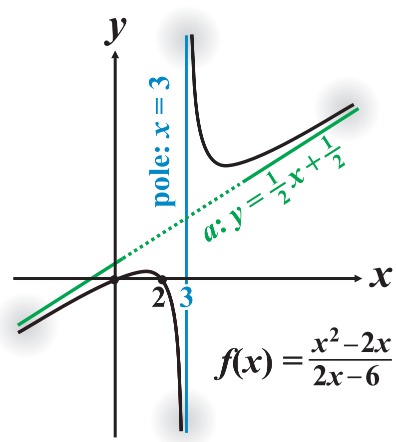
Coefficients: $a_n, b_m \neq 0$.

Degree, order of numerator: $n \in \mathbb{N}_0$. Degree, order of denominator: $m \in \mathbb{N}\backslash\{0\}$.

Properties:

▶ **Vertical asymptotes (poles)**: x_0 is called pole of f if

$y = \lim\limits_{x \to x_0} f(x) = \pm\infty$ (non- removable division by zero).



▶ **Horizontal or slant asymptote:**

Approaching function $a(x)$ for $x \to \pm\infty$. Three cases:

- $n < m$: $\lim\limits_{x \to \pm\infty} f(x) = 0 \quad \Rightarrow \quad a : y = 0$
- $n = m$: $\lim\limits_{x \to \pm\infty} f(x) = \frac{a_n}{b_m} \quad \Rightarrow \quad a : y = \frac{a_n}{b_m}$
- $n > m$: $\lim\limits_{x \to \pm\infty} f(x) = \pm\infty \quad \Rightarrow$ split

$f(x) = \frac{U(x)}{V(x)} = a(x) + \frac{u(x)}{V(x)}$ with $\lim\limits_{x \to \pm\infty} \frac{u(x)}{V(x)} = 0$

division = long division algorithm.

$\Rightarrow$ For $n = m + 1$, $a(x)$ is a slant, linear asymptote.

$\Rightarrow$ $a(x)$: Polynomial of degree $(n - m)$.

$\Rightarrow$ Limits on p. 25.

5.7 Exponential and Logarithmic Functions

▸ **Exponential functions:** $y = f(x) = a^x$ $\quad a > 0, \quad x \in \mathbb{R}.$

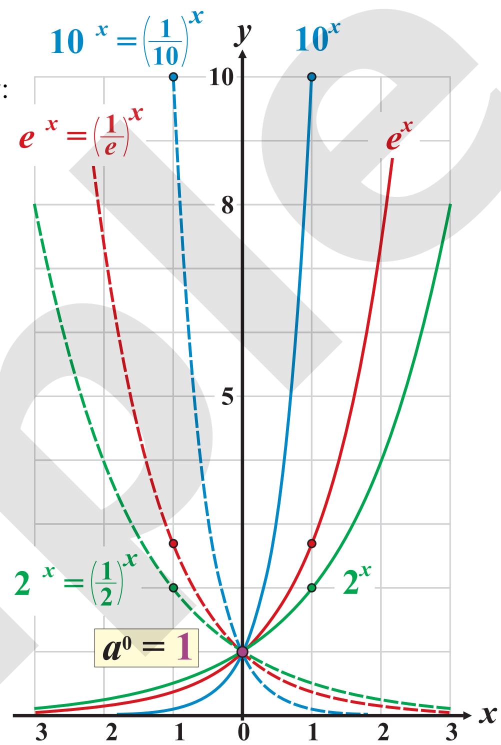
- **Euler's number:** $e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n \approx 2.718...$.
- Processes of exponential growth respectively decay:

$N(t) = N_0 \cdot a^t$ or $N(t) = N_0 \cdot e^{k\cdot t}$ where:

t	:	Time.
N_0	:	Initial population at $t = 0$.
$N(t)$	:	Population at time t.
$a = e^k$	:	Growth factor: $a = 1 + \frac{p}{100}$ with

$$p : \left\{ \begin{array}{ll} \text{growth} & (p > 0) \\ \text{decay} & (p < 0) \end{array} \right\} \text{ in } \%$$

per time unit.



⇒ Power and logarithm laws on p. 5.
⇒ Derivatives and antiderivatives on p. 29.
⇒ Limits on p. 25.

▸ **Logarithmic functions:** $\overline{f}(x) = \log_a(x)$ $\quad x > 0$; $a > 0; \quad a \neq 1.$

$\overline{f}(x) = \log_a(x)$ are inverse functions of $f(x) = a^x$:

- Common logarithm:

$\overline{f}(x) = \log_{10}(x) = \log(x)$

$\log(10^x) = x, \qquad 10^{\log(x)} = x \qquad (x > 0)$

- Natural logarithm:

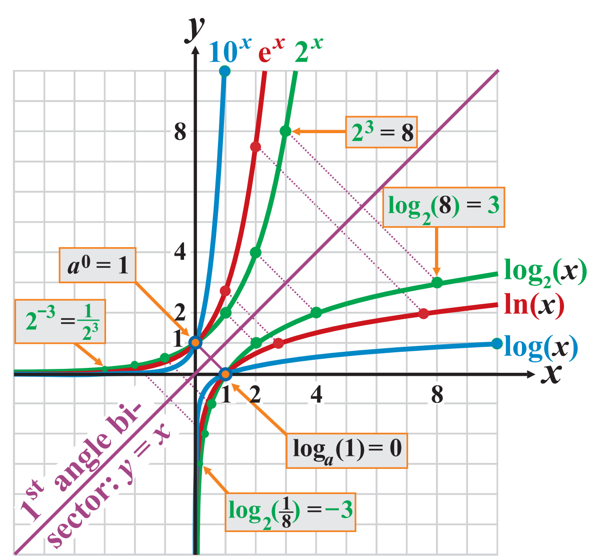
$\overline{f}(x) = \log_e(x) = \ln(x)$

$\ln(e^x) = x, \qquad e^{\ln(x)} = x \qquad (x > 0)$

- Binary logarithm:

$\overline{f}(x) = \log_2(x) = \mathrm{lb}(x)$

$\log_2(2^x) = x, \qquad 2^{\log_2(x)} = x \qquad (x > 0)$

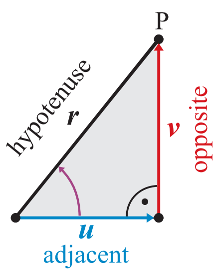


⇒ Power and logarithm laws on p. 5.
⇒ Derivatives and antiderivatives on p. 29.

5.8 Trigonometric Functions

▶ **Definition:** (see p. 6)

Right-angled triangle: $0° < \alpha < 90°$.

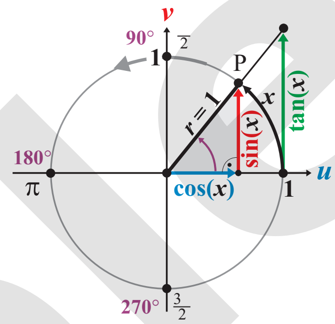


$\sin(\alpha) = \frac{v}{r} = \frac{\text{opposite}}{\text{hypotenuse}}$

$\cos(\alpha) = \frac{u}{r} = \frac{\text{adjacent}}{\text{hypotenuse}}$

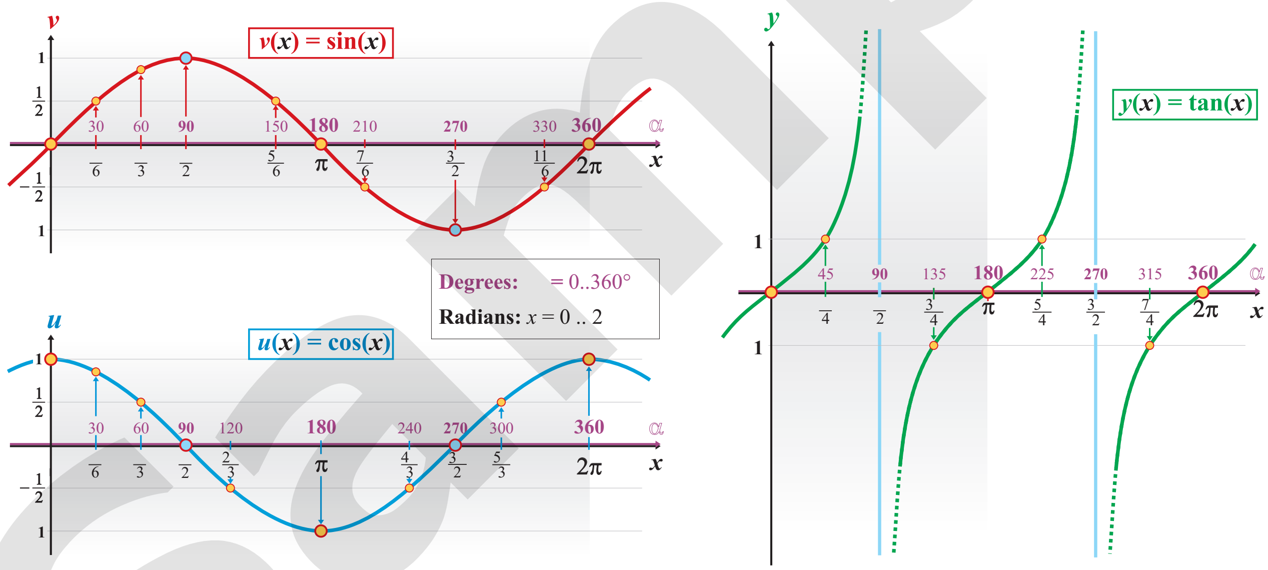
$\tan(\alpha) = \frac{v}{u} = \frac{\text{opposite}}{\text{adjacent}} = \frac{\sin(\alpha)}{\cos(\alpha)}$

Unit circle: $x,\ \alpha \in \mathbb{R}$.



▶ **Radians:** $x = \alpha \cdot \frac{\pi}{180°}$ { Length of the arc in the unit circle corresponding to the central angle α.

▶ **Graphs:**



▶ **Properties and particular values:**

	$0° \doteq 0$	$30° \doteq \frac{\pi}{6}$	$45° \doteq \frac{\pi}{4}$	$60° \doteq \frac{\pi}{3}$	$90° \doteq \frac{\pi}{2}$	Periodicity	Symmetry
$\sin(x)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$360° \doteq 2\pi$ $\sin(x+2\pi n) = \sin(x)$	$\sin(\pi - x) = \sin(x)$ $\sin(-x) = -\sin(x)$
$\cos(x)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$360° \doteq 2\pi$ $\cos(x+2\pi n) = \cos(x)$	$\cos(2\pi - x) = \cos(x)$ $\cos(-x) = \cos(x)$
$\tan(x)$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	$(\pm\infty)$	$180° \doteq \pi$ $\tan(x+\pi n) = \tan(x)$	$\tan(-x) = -\tan(x)$

▶ **Domain:** $\mathbb{D}_{\sin} = \mathbb{D}_{\cos} = \mathbb{R}$ $\quad \mathbb{D}_{\tan} = \mathbb{R}\backslash\{(\frac{\pi}{2} + n\pi),\ n \in \mathbb{Z}\}$.

▶ **Range:** $\mathbb{W}_{\sin} = \mathbb{W}_{\cos} = [-1, 1]$ $\quad \mathbb{W}_{\tan} = \mathbb{R}$.

▶ **Inverse functions:**
- $\arcsin(x)$ sometimes denoted $\sin^{-1}(x)$,
- $\arccos(x)$ sometimes denoted $\cos^{-1}(x)$,
- $\arctan(x)$ sometimes denoted $\tan^{-1}(x)$.

Derivatives and anti-derivatives on p. 29.

▶ Identities and Properties of Trigonometric Functions:

$\tan(x) = \frac{\sin(x)}{\cos(x)}$	$\sin^2(x) + \cos^2(x) = 1$	$\frac{1}{\cos^2(x)} = 1 + \tan^2(x)$
$\sin(-x) = -\sin(x)$	$\cos(-x) = \cos(x)$	$\tan(-x) = -\tan(x)$
$\sin(\pi - x) = \sin(x)$	$\cos(\pi - x) = -\cos(x)$	$\tan(\pi - x) = -\tan(x)$
$\sin(\frac{\pi}{2} \pm x) = \cos(x)$	$\cos(\frac{\pi}{2} \pm x) = \mp \sin(x)$	$\tan(\frac{\pi}{2} \pm x) = \mp \frac{1}{\tan(x)}$
$\sin(2\,x) = 2\sin(x)\cos(x)$	$\cos(2\,x) = \begin{cases} 2\cos^2(x) - 1 \\ \cos^2(x) - \sin^2(x) \\ 1 - 2\sin^2(x) \end{cases}$	$\tan(2\,x) = \frac{2\,\tan(x)}{1 - \tan^2(x)}$
$\sin(3\,x) = 3\sin(x) - 4\sin^3(x)$	$\cos(3\,x) = 4\cos^3(x) - 3\cos(x)$	$\tan(3\,x) = \frac{3\tan(x) - \tan^3(x)}{1 - 3\,\tan^2(x)}$
$\sin^2(\frac{x}{2}) = \frac{1 - \cos(x)}{2}$	$\cos^2(\frac{x}{2}) = \frac{1 + \cos(x)}{2}$	$\tan^2(\frac{x}{2}) = \frac{1 - \cos(x)}{1 + \cos(x)}$

$\sin(x \pm y) = \sin(x) \cdot \cos(y) \pm \cos(x) \cdot \sin(y)$	$\tan(x + y) = \frac{\tan(x) + \tan(y)}{1 - \tan(x) \cdot \tan(y)}$
$\cos(x \pm y) = \cos(x) \cdot \cos(y) \mp \sin(x) \cdot \sin(y)$	$\tan(x - y) = \frac{\tan(x) - \tan(y)}{1 + \tan(x) \cdot \tan(y)}$
$\sin(x) + \sin(y) = 2 \cdot \sin\left(\frac{x+y}{2}\right) \cdot \cos\left(\frac{x-y}{2}\right)$	$\sin(x) - \sin(y) = 2 \cdot \cos\left(\frac{x+y}{2}\right) \cdot \sin\left(\frac{x-y}{2}\right)$
$\cos(x) + \cos(y) = 2 \cdot \cos\left(\frac{x+y}{2}\right) \cdot \cos\left(\frac{x-y}{2}\right)$	$\cos(x) - \cos(y) = -2 \cdot \sin\left(\frac{x+y}{2}\right) \cdot \sin\left(\frac{x-y}{2}\right)$

⇒ Derivatives and antiderivatives see p. 29.

6 Equations

6.1 Fundamental Theorem of Algebra

Every polynomial of degree n can be written as a product of $k \leq n$ linear factors and irreducible quadratic factors $q(x) \neq 0$ for all $x \in \mathbb{R}$:

Equation in normal form:

$a_n x^n + a_{n-1} x^{n-1} + \ldots. + a_1 x + a_0 = 0$

factorizing →
← **expanding**

Decomposition in linear factors:

$a_n \cdot (x - x_1) \cdot (x - x_2) \cdot \ldots \cdot (x - x_k) \cdot q(x) = 0$

⇒ roots (solutions) x_1, x_2, ..., x_k.

6.2 Quadratic Equations

$a \cdot x^2 + b \cdot x + c = 0 \quad a, b, c \in \mathbb{R}, \quad a \neq 0.$

Discriminant: $D = b^2 - 4 \cdot a \cdot c$

Solutions: $x_{1,2} = \dfrac{-b \pm \sqrt{b^2 - 4 \cdot a \cdot c}}{2 \cdot a}$ $\quad D \geq 0.$

Viète's formulas:

Product of solutions: $x_1 \cdot x_2 = \dfrac{c}{a}$

Sum of solutions: $x_1 + x_2 = -\dfrac{b}{a}$

⇒ Quadratic functions on p. 17.

6.3 Polynomial Equations of 3^{rd} and Higher Degree

$a \cdot x^3 + b \cdot x^2 + c \cdot x + d = 0 \quad a, b, c, d \in \mathbb{R}, a \neq 0.$

Method: Normalize to $a = 1$ (division by $a \neq 0$), that is $x^3 + b' \cdot x^2 + c' \cdot x + d' = 0$.
If there is an integer solution x_1, it must be a divisor of d'. Find solution x_1 by trying the divisors of d'. Then divide the equation by $(x - x_1)$ and find further solutions.

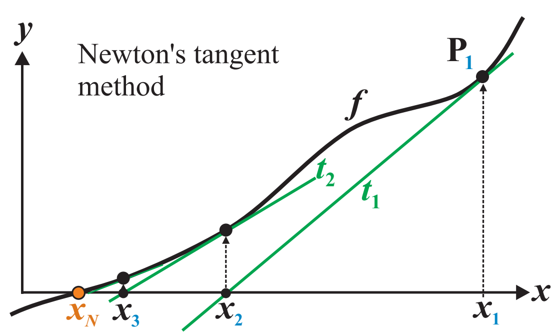
6.4 Numerical Methods to Calculate Zeros

To calculate a zero N(x_N / 0) of a function $f(x)$, start with a guess x_1. Then, set up a recursive sequence x_1, x_2, x_3,... with limit x_N.

- **Secant method:**

  Choose P(a / $f(a)$) and Q(b / $f(b)$) with $f(a) \cdot f(b) < 0$.
  Unsing $x_1 = a$ as initial value, calculate:

  $$x_{n+1} = x_n - f(x_n)\,\frac{b - x_n}{f(b) - f(x_n)} \underset{n \to \infty}{\longrightarrow} x_N$$



- **Newton's tangent method:**

  Choose $P_1(x_1 \ / \ f(x_1))$ with $f'(x_1) \neq 0$. Then:

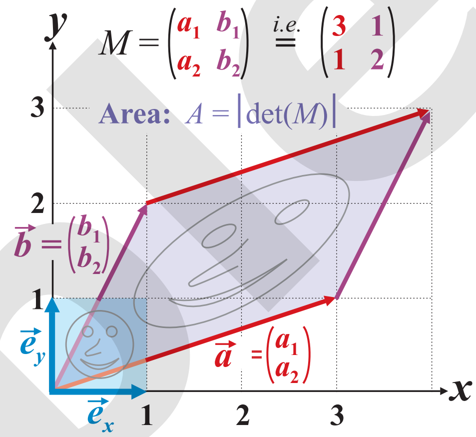
  $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \underset{n \to \infty}{\longrightarrow} x_N$$

  The sequence is not always convergent.

7 Matrices, Systems of Linear Equations

7.1 Simultaneous Linear Equations, 2 × 2 Matrices

$$\left| \begin{array}{l} a_1 x + b_1 y = c_1 \\ a_2 x + b_2 y = c_2 \end{array} \right| \quad \Rightarrow \quad \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \quad \text{briefly: } \boxed{M \cdot \vec{x} = \vec{c}}$$



- Multiplying the matrix M from the left by its inverse M^{-1}, solves the equation $M \cdot \vec{x} = \vec{c}$ for $\vec{x}$: $\boxed{\vec{x} = M^{-1} \cdot \vec{c}}$ (if M^{-1} exists).
- The Matrix $M = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$ represents a linear transformation from $\mathbb{R}^2 \to \mathbb{R}^2$:
  Each vector $\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ is assigned
  the vector $\vec{c} = \begin{pmatrix} c_x \\ c_y \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$.
- The **columns** $\vec{a}$, $\vec{b}$ of the matrix M are the **images of the Cartesian** unit vectors $\vec{e}_x$ and $\vec{e}_y$ under the linear transformation M.

7.2 Operations and Properties of Matrices

▶ **Identity matrices:** $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\quad I_n = \begin{pmatrix} 1 & \cdots & & 0 \\ \vdots & 1 & & \\ & & \ddots & \\ 0 & \cdots & & 1 \end{pmatrix}$

- $M \cdot I_n = I_n \cdot M = M$

▶ **Addition:** $\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \pm \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} = \begin{pmatrix} a_1 \pm u_1 & b_1 \pm v_1 \\ a_2 \pm u_2 & b_2 \pm v_2 \end{pmatrix}$

- $M_1 + M_2 = M_2 + M_1$
- $(M_1 + M_2) + M_3 = M_1 + (M_2 + M_3)$

▶ **Multiplication by a scalar (real number):** $k \cdot \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = \begin{pmatrix} k\,a_1 & k\,b_1 \\ k\,a_2 & k\,b_2 \end{pmatrix}; \quad k \in \mathbb{R}$.

▶ **Multiplication by a vector:** $\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_1 x + b_1 y \\ a_2 x + b_2 y \end{pmatrix}$

▶ **Product of two matrices:** $\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \cdot \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} = \begin{pmatrix} a_1 u_1 + b_1 u_2 & a_1 v_1 + b_1 v_2 \\ a_2 u_1 + b_2 u_2 & a_2 v_1 + b_2 v_2 \end{pmatrix}$

- $M_1 \cdot M_2 \neq M_2 \cdot M_1$
- $(M_1 \cdot M_2) \cdot M_3 = M_1 \cdot (M_2 \cdot M_3)$

▶ **Transposition:** $M^T = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}^T = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$ $\quad M^T = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}^T = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$

- $(M_1 + M_2)^T = M_1^T + M_2^T$
- $(M_1 \cdot M_2)^T = M_2^T \cdot M_1^T$
- $(M^T)^T = M$

▶ **Inverse matrix:** $M \cdot M^{-1} = M^{-1} \cdot M = I_n$

$M^{-1} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}^{-1} = \frac{1}{\det(M)} \begin{pmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{pmatrix}$ if $\det(M) \neq 0$.

In general: $[M \mid E_n] \overset{\text{Gauss}}{\longrightarrow} [E_n \mid M^{-1}]$.

- $(M_1 \cdot M_2)^{-1} = M_2^{-1} \cdot M_1^{-1}$
- $(M^{-1})^{-1} = M$
- $(M^{-1})^T = (M^T)^{-1}$

▶ **Determinant:** $\det \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = a_1 \cdot b_2 - a_2 \cdot b_1$

$$\det \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} = a_1 \cdot \det \begin{pmatrix} b_2 & c_2 \\ b_3 & c_3 \end{pmatrix} - b_1 \cdot \det \begin{pmatrix} a_2 & c_2 \\ a_3 & c_3 \end{pmatrix} + c_1 \cdot \det \begin{pmatrix} a_2 & b_2 \\ a_3 & b_3 \end{pmatrix}$$

- $\det(A \cdot B) = \det(A) \cdot \det(B)$
- $\det(I_n) = 1$
- $\det(A^T) = \det(A) \qquad \det(A^{-1}) = \frac{1}{\det(A)}$
- $\det(k \cdot A) = k^n \cdot \det(A)$

▶ **Rank:** Number of linearly independent rows (or columns) of a matrix. M is called a...

- *regular* $n \times n$ matrix, if: $\det(M) \neq 0 \Leftrightarrow \text{rank}(M) = n \Leftrightarrow M^{-1}$ exists.
- *singular* $n \times n$ matrix, if: $\det(M) = 0 \Leftrightarrow \text{rank}(M) < n \Leftrightarrow M^{-1}$ does not exist.

▶ **Orthogonal matrices:** $M \cdot M^T = M^T \cdot M = I$ respectively $M^T = M^{-1}$ holds.

▶ **Eigenvectors, eigenvalues:** A vector $\vec{v}$ is called eigenvector of M to the eigenvalue λ, if $M \cdot \vec{v} = \lambda \cdot \vec{v}$ holds. The linear function M leaves the orientation of $\vec{v}$ **un**changed.

[i] Calculate the eigenvalues λ: $\quad \det(M - \lambda \cdot E_n) = 0 \quad \Rightarrow \quad \lambda_1, \lambda_2, ...$

[ii] Solve for the eigenvectors $\vec{v}_k$: $\quad (M - \lambda_k \cdot E_n) \cdot \vec{v}_k = 0 \quad \Rightarrow \quad \vec{v}_1, \vec{v}_2, ...$

▶ **Elementary matrix operations (Gaussian algorithm):**

- Multiplying a row by a real number $k \neq 0$.
- Adding a row to another row.
- Switching two rows of a matrix.

Solving systems of linear equations:

- Write simultaneous linear equations as a matrix $[M \mid \vec{c}\,]$.
- Transform M to unity matrix using the Gaussian algorithm.

▶ **Special matrices:**

$\begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}$ rotates $\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ by the angle α anticlockwise about O(0 / 0).

Reflection of $\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ about the... x-axis: $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad y$-axis: $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

Matrices
Gauss

please cut out this part

8 Sequences and Series

Definition: A **sequence** is a **function** $u : \mathbb{N} \to \mathbb{R}, \quad n \mapsto u_n$. (See. p. 14.) Notation:

- **Explicit formula:** $u_n = \{\text{formula in } n \text{ only}\}$.
- **Recurrence formula:** $u_{n+1} = \{\text{formula in } u_n, u_{n-1}, \dots\}$ with given initial value u_1.

A **series** $s_1, s_2, s_3, \dots$ is the sequence of partial sums of a given sequence $\{u_n\}_{n \in \mathbb{N}}$:

$$s_1 = u_1 \xrightarrow[+u_2]{} s_2 = u_1 + u_2 \xrightarrow[+u_3]{} s_3 = u_1 + u_2 + u_3 \quad \dots \quad \boxed{s_n = \sum_{k=1}^{n} u_k}$$

8.1 Arithmetic Sequences and Series

Constant difference $\boxed{d = u_2 - u_1}$ between consecutive elements: $\quad u_1 \xrightarrow[+d]{} u_2 \xrightarrow[+d]{} u_3 \xrightarrow[+d]{} \dots$

	Recurrence	Explicit formula
Sequence	$u_{n+1} = u_n + d$	$\boxed{u_n = u_1 + (n-1) \cdot d}$
Series	$s_{n+1} = s_n + u_{n+1}$	$\boxed{s_n = \frac{n}{2} \cdot (u_1 + u_n) = \frac{n}{2} \cdot (2u_1 + (n-1) \cdot d)}$

8.2 Geometric Sequences and Series

Constant ratio (quotient) $\boxed{r = \frac{u_2}{u_1}}$ between consecutive elements: $\quad u_1 \xrightarrow[\cdot r]{} u_2 \xrightarrow[\cdot r]{} u_3 \xrightarrow[\cdot r]{} \dots$

	Recurrence	Explicit formula
Sequence	$u_{n+1} = u_n \cdot r$	$\boxed{u_n = u_1 \cdot r^{n-1}}$
Series	$s_{n+1} = s_n + u_{n+1}$	$\boxed{s_n = u_1 \cdot \frac{1-r^n}{1-r}}$ $r \neq 1, \quad s_n = n \cdot u_1$ if $r = 1$.
		$\boxed{s = \lim_{n\to\infty} s_n = \frac{u_1}{1-r}}$ if $\lvert r \rvert < 1$. Limits on p. 25.

8.3 Other Series

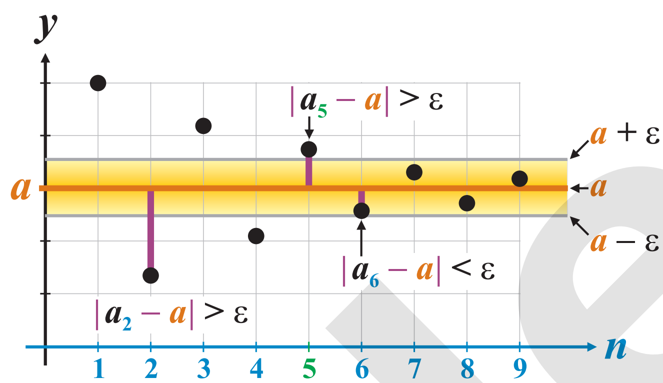
$$\sum_{k=1}^{n} \frac{1}{k^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2} \xrightarrow[n\to\infty]{} \frac{\pi^2}{6}$$

$$\sum_{k=1}^{n} \frac{1}{k} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} \xrightarrow[n\to\infty]{} \infty \qquad \text{(Harmonic series)}$$

$$\sum_{k=1}^{n} k = \frac{1}{2} \cdot n \cdot (n+1) \qquad \sum_{k=1}^{n} k^2 = \frac{n}{6} \cdot (n+1) \cdot (2n+1) \qquad \sum_{k=1}^{n} k^3 = \left(\frac{1}{2} \cdot n \cdot (n+1)\right)^2$$

8.4 Limits

A sequence is called **convergent** with limit $a = \lim_{n\to\infty} a_n$, if for **any** arbitrarily small number $\varepsilon > 0$ there is an index $N \in \mathbb{N}$, such that $|a_n - a| < \varepsilon$ holds for all $n > N$.
For arbitrarily large n, the distance between a_n and a tends to 0.



- A limit is always **unique** and **finite**.
- Sequences without limit (or such with $\lim_{n\to\infty} a_n = \pm\infty$) are called **divergent**.
- **Undefined expressions:** $\frac{0}{0}$, $\frac{(\pm\infty)}{(\pm\infty)}$, $0 \cdot (\pm\infty)$ and $\infty - \infty$

▶ **Limit identities:** Assuming that $a = \lim_{n\to\infty} a_n$ and $b = \lim_{n\to\infty} b_n$ exist:

- $\lim_{n\to\infty} (a_n \pm b_n) = a \pm b$
- $\lim_{n\to\infty} (c \cdot a_n) = c \cdot a$
- $\lim_{n\to\infty} (a_n \cdot b_n) = a \cdot b$
- $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}$ if $b \neq 0$

⇒ Similar identities hold for limits $\lim_{x\to x_0} f(x)$.

▶ **Particular limits:**

- $\lim_{n\to\pm\infty} \frac{1}{n} = 0$
- $\lim_{x\to\infty} a^x = \begin{cases} 0, & \text{if } -1 < a < 1 \\ 1, & \text{if } a = 1 \\ \infty, & \text{if } a > 1 \end{cases}$
- $\lim_{n\to\pm 0} \frac{1}{n} = \pm\infty$
- $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$
- $\lim_{x\to\infty} \frac{a_n\,x^n + a_{n-1}\,x^{n-1} + \ldots + a_1\,x + a_0}{b_m\,x^m + b_{m-1}\,x^{m-1} + \ldots + b_1\,x + b_0} = \begin{cases} 0, & n < m \\ \frac{a_n}{b_m}, & n = m \\ \pm\infty, & n > m \end{cases}$
- $\lim_{n\to\infty} \frac{a^n}{n!} = 0$
- $\lim_{x\to 0} \frac{e^x - 1}{x} = 1.$

▶ **Order of convergence (dominance rule):**

Exponential growth is faster than power growth: $\lim_{x\to\infty} \frac{x^n}{e^x} = 0$

Power growth is faster than logarithmic growth: $\lim_{x\to\infty} \frac{\ln(x)}{x^n} = 0$ $\quad n > 0.$

▶ **L'Hôpital's rule**: Assume $\lim_{x\to x_0} f(x) = 0$ (or ∞) **and** $\lim_{x\to x_0} g(x) = 0$ (or ∞), then:

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}$$

Example: $\lim_{x\to 0} \frac{\sin(x)}{x} = \lim_{x\to 0} \frac{\cos(x)}{1} = 1.$

8.5 Mean Values

Let x_1, x_2 ,..., x_n be n given values.

Arithmetic mean value	$\overline{x}_{\mathrm{A}} = \frac{x_1 + x_2 + ... + x_n}{n}$ (see p. 39)
Weighted average value (see expected value, p. 37)	$\overline{x}_{\mathrm{A}} = \frac{p_1 \cdot x_1 + p_2 \cdot x_2 + ... + p_n \cdot x_n}{p_1 + p_2 + ... + p_n}$ where p_1, p_2, ..., p_n are the relative frequencies of the values x_1, x_2 ,..., x_n.
Root mean square value	$\overline{x}_{\mathrm{RMS}} = \sqrt{\frac{x_1^2 + x_2^2 + ... + {x_n}^2}{n}}$
Geometric mean value	$\overline{x}_{\mathrm{G}} = \sqrt[n]{x_1 \cdot x_2 \cdot \ldots \cdot x_n}$
Harmonic mean value	$\overline{x}_{\mathrm{H}} = n \cdot \left(\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n} \right)^{-1}$ $x_k \neq 0$.

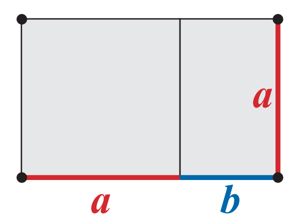
Inequalities: $\overline{x}_{\mathrm{H}} \leq \overline{x}_{\mathrm{G}} \leq \overline{x}_{\mathrm{A}} \leq \overline{x}_{\mathrm{RMS}}$ hold, if $x_k \geq 0$ for all $k = 1,\ 2,\ \ldots\ n$.

8.6 Harmonic Section, Golden Ratio

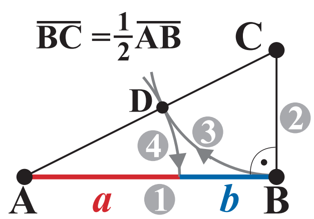
Two lines are in the Golden Ratio Φ if they intersect in the Harmonic Ratio: $\Phi = \frac{a}{b} = \frac{a+b}{a}$ therefore:

$$\Phi^2 - \Phi - 1 = 0 \quad \Rightarrow \quad \Phi_{1,2} = \frac{1 \pm \sqrt{5}}{2} = \begin{cases} \Phi = 1.618... \\ \overline{\Phi} = -0.618... \end{cases}$$

Golden rectangle:



Harmonic section of $\overline{\mathrm{AB}}$:



Properties:

- $\overline{\Phi} = -\frac{1}{\Phi}$
- Φ is irrational and can also be written as:

$$\Phi = \sqrt{1 + \sqrt{1 + \sqrt{1 + ...}}} \qquad \Phi = 1 + \frac{1}{1 + \frac{1}{1 + ...}}$$

8.7 Mathematical Induction

Method of mathematical proof for statements $\mathbb{A}_n$ on natural numbers ($\mathbb{N}$).

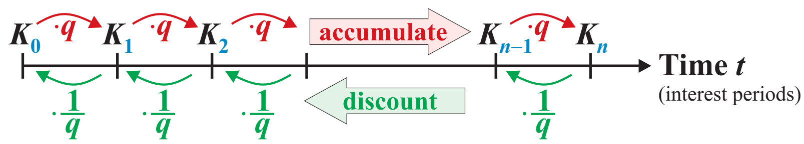
(I) **Base clause:** Show that $\mathbb{A}_1$ is true.
Note: Instead of $n = 1$ a different initial value n_0 can be chosen. The proof holds for all $n \geq n_0$.

(II) **Recursive clause, step from n to $(n+1)$:** Calculate $\mathbb{A}_{n+1}$ recursively and show that the result coincides with the one calculated directly, that is $\mathbb{A}_{n+1}$ obtained by substituting n by $(n+1)$ in $\mathbb{A}_n$.

9 Financial Mathematics

Interest factor: $q = 1 + \frac{p}{100} = 1 + i$ p : Interest (p.a.) in %, $i = \frac{p}{100}$: Interest rate.

▶ Capital with compound interest: Seed capital K_0, duration n years:



Accumulated value:

$K_n = K_0 \cdot q^n$

Cash value:

$K_0 = K_n \cdot \frac{1}{q^n}$

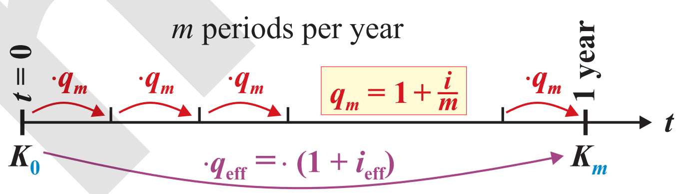
▶ Interest for parts of a year:

Linear:	**Compound interest:**	**Continuous interest:**
Capital K_T after T days of simple interest:	m interest periods per year, duration: n years.	Interest is paid continuously:
$K_T = K_0 + K_0 \cdot i \cdot \frac{T}{360}$	$K_{n\cdot m} = K_0 \cdot \left(1 + \frac{i}{m}\right)^{n\cdot m}$	$K_\infty = \lim\limits_{m\to\infty} K_{n\cdot m} = K_0 \cdot e^{i\cdot n}$

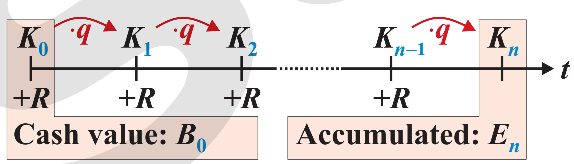
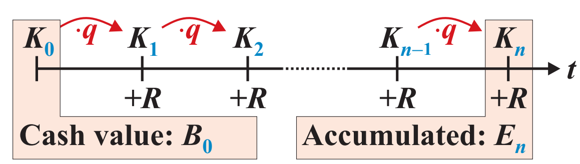
Effective rate of interest:

$i_{\text{eff}} = \left(1 + \frac{i}{m}\right)^m - 1$

$q_m = \sqrt[m]{q_{\text{eff}}}$



▶ Rent computation (annuities): n rents R are paid to an initial capital K_0:

Rent R paid in advance:		**Rent R paid in arrear:**	
			
Cash value B_0 = $K_0 + \frac{R}{q^{n-1}}\,\frac{q^n-1}{q-1}$	**Accumulated value E_n =** $K_0\, q^n + R\, q\,\frac{q^n-1}{q-1}$	**Cash value B_0 =** $K_0 + \frac{R}{q^n}\,\frac{q^n-1}{q-1}$	**Accumulated value E_n =** $K_0\, q^n + R\,\frac{q^n-1}{q-1}$

⇒ The rent R is also called amortization rate or annuity.

▶ Derivatives in economics:

Marginal function: $f'(x) = \frac{df}{dx}$

Growth rate: $r(t) = \frac{f'(t)}{f(t)} = \frac{d}{dt}\ln(f(t))$

Elasticity: $\varepsilon_f(x) = x \cdot \frac{f'(x)}{f(x)}$

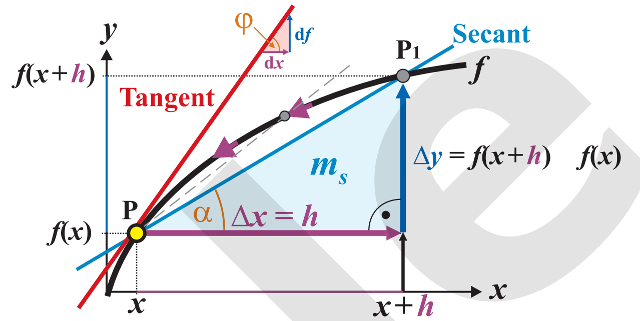
Mean val.
Finance

please cut out

10 Differential Calculus

- **Slope of the secant, difference quotient:**
Average rate of change (slope) of $f(x)$ in the interval $[x,\ x+h]$:

$$m_s = \frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{h} = \tan(\alpha)$$



- **Slope of the tangent, differential quotient:**
Instantaneous rate of change, gradient of $f(x)$ at $P(x\ /\ f(x))$:
Definition of the first derivative:

$$m_t = f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \tan(\varphi)$$

Slope on p. 16, limits on p. 25.

10.1 Rules of Differentiation

Let $f(x)$, $u(x)$ and $v(x)$ be differentiable functions and $c \in \mathbb{R}$ a constant.

- **Constant summand:** $f(x) = u(x) \pm c$

$$f'(x) = u'(x) \pm 0$$

- **Constant factor:** $f(x) = c \cdot u(x)$

$$f'(x) = c \cdot u'(x)$$

- **Sum rule:** $f(x) = u(x) \pm v(x)$

$$f'(x) = u'(x) \pm v'(x)$$

- **Product rule:** $f(x) = u(x) \cdot v(x)$

$$f'(x) = u'(x) \cdot v(x) + u(x) \cdot v'(x)$$

- **Quotient rule:** $f(x) = \frac{u(x)}{v(x)}$

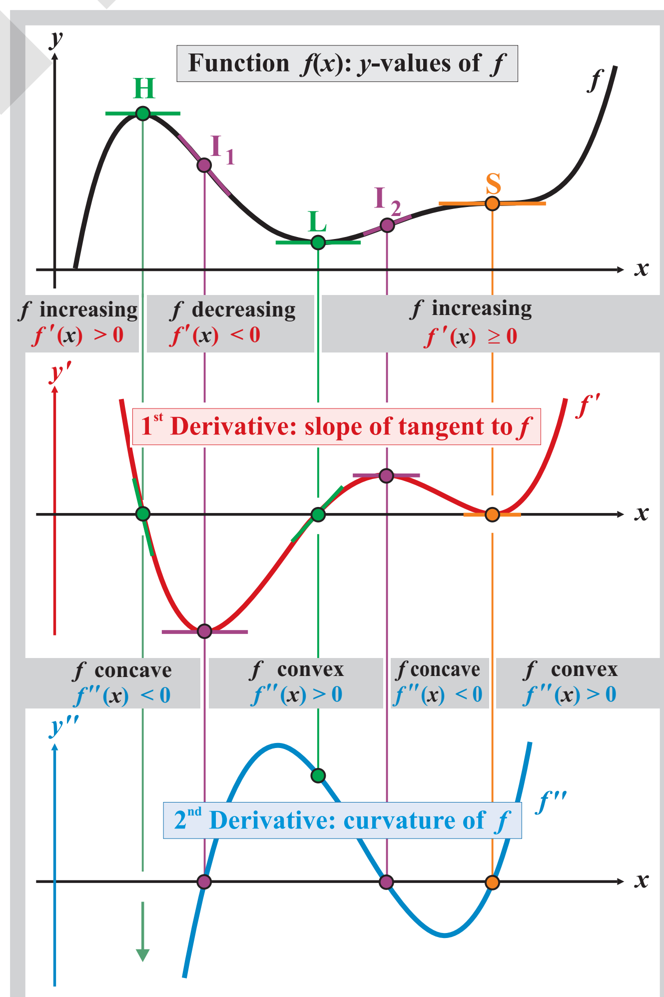
$$f'(x) = \frac{u'(x) \cdot v(x) - u(x) \cdot v'(x)}{(\,v(x)\,)^2}$$

- **Chain rule:** $f(x) = u(v(x))$

$$f'(x) = u'(v) \cdot v'(x) = \frac{du}{dv} \cdot \frac{dv}{dx}$$

"outer derivative times inner derivative"

Stationary points and points of inflection, relationship between $f(x)$, $f'(x)$ and $f''(x)$:

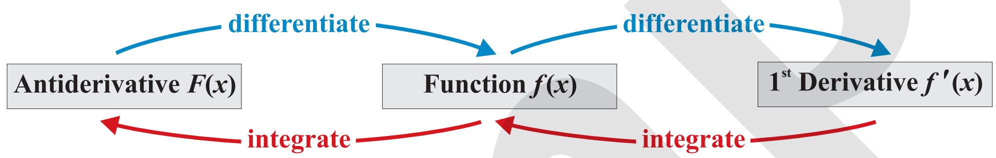


10.2 Sufficient Criteria to Calculate Particular Points

		f	f'	f''	f'''
Zero	$\mathrm{N}(x_N \ / \ 0)$	$f(x_N) = 0$	-	-	-
High point	$\mathrm{H}(x_H \ / \ f(x_H))$		$f'(x_H) \overset{\bigstar}{=} 0$	$f''(x_H) \overset{\blacklozenge}{<} 0$	-
Low point	$\mathrm{L}(x_L \ / \ f(x_L))$		$f'(x_L) \overset{\bigstar}{=} 0$	$f''(x_L) \overset{\blacklozenge}{>} 0$	-
Stationary point of inflection	$\mathrm{S}(x_S \ / \ f(x_S))$		$f'(x_S) \overset{\bigstar}{=} 0$	$f''(x_S) \overset{\bigstar}{=} 0$	$f'''(x_S) \overset{\blacklozenge}{\neq} 0$
Inflection point	$\mathrm{I}(x_I \ / \ f(x_I))$		-	$f''(x_I) \overset{\bigstar}{=} 0$	$f'''(x_I) \overset{\blacklozenge}{\neq} 0$

$\bigstar$ = necessary condition. $(\bigstar + \blacklozenge)$ = sufficient condition.

10.3 Table of Derivatives and Antiderivatives (Primitives)



$\frac{x^{n+1}}{n+1} \quad [n \neq -1]$	x^n	$n \cdot x^{n-1}$
$\ln\|x\|$	$\frac{1}{x} = x^{-1}$	$-x^{-2} = -\frac{1}{x^2}$
$\frac{2}{3} \cdot x^{\frac{3}{2}}$	$\sqrt{x} = x^{\frac{1}{2}}$	$\frac{1}{2 \cdot \sqrt{x}}$
e^x	e^x	e^x
$x \cdot (\ln\|x\| - 1)$	$\ln\|x\|$	$\frac{1}{x} = x^{-1}$
$\frac{1}{\ln(a)} \cdot a^x$	a^x	$a^x \cdot \ln(a)$
$\frac{x}{\ln(a)} \cdot (\ln\|x\| - 1)$	$\log_a\|x\|$	$\frac{1}{x \cdot \ln(a)}$
Note: Variable x in radians! Trigonometric functions on p. 19, 20.		
$-\cos(x)$	$\sin(x)$	$\cos(x)$
$\sin(x)$	$\cos(x)$	$-\sin(x)$
$-\ln(\|\cos(x)\|)$	$\tan(x)$	$\frac{1}{\cos^2(x)} = 1 + \tan^2(x)$
$x \cdot \arcsin(x) + \sqrt{1 - x^2}$	$\arcsin(x)$	$\frac{1}{\sqrt{1-x^2}}$
$x \cdot \arccos(x) - \sqrt{1 - x^2}$	$\arccos(x)$	$-\frac{1}{\sqrt{1-x^2}}$
$x \cdot \arctan(x) - \frac{\ln(x^2+1)}{2}$	$\arctan(x)$	$\frac{1}{x^2+1}$

11 Integral Calculus

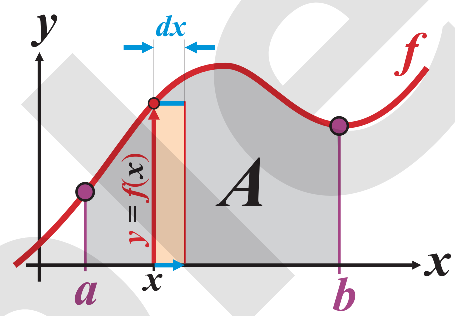
Let $F(x)$ be an **antiderivative (primitive)** of $f(x)$, that is a function satisfying $F'(x) = f(x)$. Then, any further antiderivative $F_1(x)$ of $f(x)$ may differ by an additive constant only: $F_1(x) = F(x) + C$. The constant C is called **constant of integration**.

- **Indefinite integral: Set of all antiderivatives:**

  $\int f(x)\,dx = \{F(x) + C \mid C \in \mathbb{R}\}$ **with** constant C.

- **Definite integral, Fundamental Theorem of Calculus:**

  $$A = \int_a^b f(x)\,dx = F(b) - F(a) = [F(x)]_a^b$$



$|A|$: Area under $f(x)$ and the x-axis between the integration limits $x = a$ and $x = b$ if $f(x) \neq 0$ for all $x \in [a,\ b]$.

11.1 Rules of Integration

▶ **Constant rule:** $$\int_a^b (\,c \cdot f(x)\,)\,dx = c \cdot \int_a^b f(x)\,dx$$

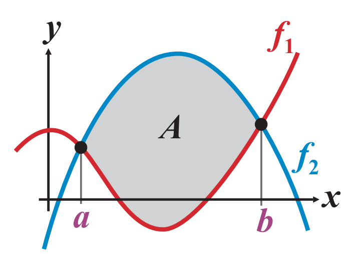
▶ **Sum rule:** $$\int_a^b (\,u(x) \pm v(x)\,)\,dx = \int_a^b u(x)\,dx \pm \int_a^b v(x)\,dx$$

▶ **Orientation of integral:** $$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

▶ **Interval additivity:** $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

▶ **Signed area:** $$\left.\begin{array}{l} f(x) \geq 0 \text{ for } x \in [a,\ b] \\ f(x) \leq 0 \text{ for } x \in [a,\ b] \end{array}\right\} \Rightarrow \int_a^b f(x)\,dx \begin{cases} \geq 0 \\ \leq 0 \end{cases}$$

▶ **Area between f_1 and f_2:** $$A = \int_a^b |\,f_2(x) - f_1(x)\,|\,dx$$



▶ **Integration by parts:**

$$\int_a^b u(x) \cdot v'(x)\,dx = [\,u(x) \cdot v(x)\,]_a^b - \int_a^b u'(x) \cdot v(x)\,dx$$

▶ **Substitution rule:** Let $f(x) = u(\,v(x)\,)$ be a composite function. $U(\,v\,)$ denotes an anti-derivative of the outer function. Then: $$\int_a^b u(v(x)) \cdot v'(x)\,dx = \int_{v(a)}^{v(b)} u(v)\,dv = [\,U(\,v\,)\,]_{v(a)}^{v(b)}$$

11.2 Volume of a Solid of Revolution and Arc Length

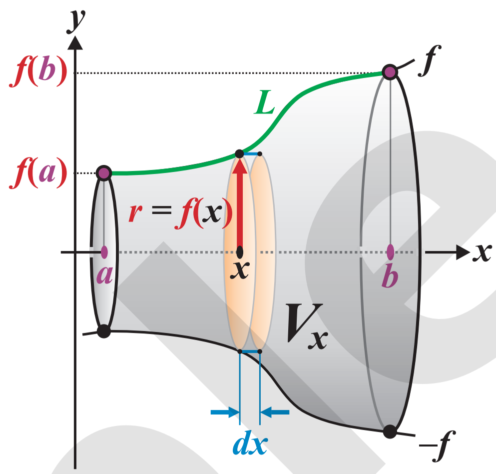
- **Rotation about x-axis:** $V_x = \pi \int_a^b (\, f(x)\,)^2 \, dx$

  generalization on p. 13.

- **Rotation about y-axis:** $V_y = \pi \int_{f(a)}^{f(b)} (\,\overline{f}(y)\,)^2 \, dy$

  $y = f(x)$ strictly monotone.
  $x = \overline{f}(y)$ is the inverse function of $y = f(x)$.
  $\Rightarrow$ Inverse functions on p. 14.

- **Arc length:** $L = \int_a^b \sqrt{1 + (\, f'(x)\,)^2} \, dx$



11.3 Power Series, Taylor Polynomials

Taylor polynomial $T_n(x)$: Approximation of a function $f(x)$ at x_0 by a polynomial of n^{th} degree:

$$T_n(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x_0)\,(x - x_0)^k$$

where $f^{(k)}(x)$ denotes the k^{th} derivative of $f(x)$. In detail:

$$T_n(x) = f(x_0) + f'(x_0)\,(x - x_0) + \frac{1}{2!} f''(x_0)\,(x - x_0)^2 + \ldots + \frac{1}{n!} f^{(n)}(x_0)\,(x - x_0)^n$$

Remainder term: $R_n(x) = f(x) - T_n(x) = \frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n+1)}(x_0 + \alpha(x - x_0)), \quad 0 < \alpha < 1.$

Power Series:

Term	Power series	Valid for
$(1+x)^n$	$1 + \binom{n}{1} x + \binom{n}{2} x^2 + \binom{n}{3} x^3 + \ldots$	$n \in \mathbb{N};\, \lvert x\rvert < 1$
$\frac{1}{1+x}$	$1 - x + x^2 - x^3 \pm \ldots$	$\lvert x\rvert < 1$
$\sqrt{1+x}$	$1 + \frac{1}{2} x - \frac{1}{2\cdot 4} x^2 + \frac{1\cdot 3}{2\cdot 4\cdot 6} x^3 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8} x^4 \pm \ldots$	$\lvert x\rvert < 1$
e^x	$1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \frac{1}{4!} x^4 + \ldots$	$x \in \mathbb{R}$
$\ln(x)$	$(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 \mp \ldots$	$0 < x \le 2$
$\sin(x)$	$x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \frac{1}{7!} x^7 \pm \ldots$	$x \in \mathbb{R}$
$\cos(x)$	$1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 - \frac{1}{6!} x^6 \pm \ldots$	$x \in \mathbb{R}$
$\tan(x)$	$x + \frac{1}{3} x^3 + \frac{2}{15} x^5 + \frac{17}{315} x^7 + \ldots$	$\lvert x\rvert < \frac{\pi}{2}$
$\arcsin(x)$	$x + \frac{1}{2\cdot 3} x^3 + \frac{1\cdot 3}{2\cdot 4\cdot 5} x^5 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7} x^7 + \ldots$	$\lvert x\rvert \le 1$
$\arctan(x)$	$x - \frac{1}{3} x^3 + \frac{1}{5} x^5 - \frac{1}{7} x^7 \pm \ldots$	$\lvert x\rvert < 1$

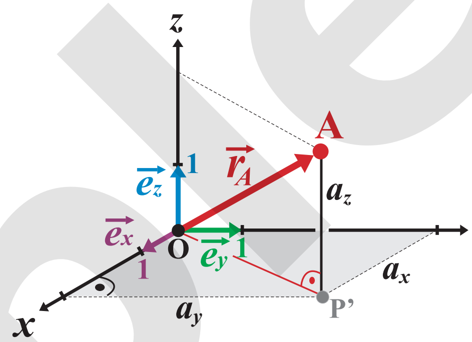
12 Vector Geometry

Definition: A vector $\vec{r}_A$ describes a **translation or displacement** (from O to A).

Vectors have a length (= magnitude, absolute value) and an orientation (direction). Vectors can be parallely shifted: Vectors do not have a fixed initial point.

▶ **Standard unit vectors:**

$$\vec{e}_x = \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad \vec{e}_y = \begin{pmatrix}0\\1\\0\end{pmatrix}, \quad \vec{e}_z = \begin{pmatrix}0\\0\\1\end{pmatrix}.$$



▶ **Linear combination:** every 3-dimensional vector $\vec{r}_A$ can be written as a linear combination of $\vec{e}_x$, $\vec{e}_y$, $\vec{e}_z$:

$$\vec{r}_A = \begin{pmatrix}a_x\\a_y\\a_z\end{pmatrix} = a_x \cdot \vec{e}_x + a_y \cdot \vec{e}_y + a_z \cdot \vec{e}_z.$$

a_x, a_y, a_z are called the components of $\vec{r}_A$.

▶ **Magnitude, length:** $|\vec{r}_A| = r_A = \overline{\mathrm{OA}} = \sqrt{a_x^2 + a_y^2 + a_z^2}$ distance from O to A.

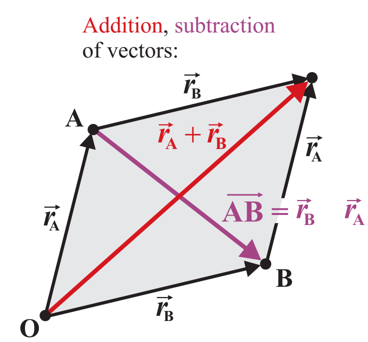
▶ **Position vector** of $A(a_x, a_y, a_z)$: $\vec{r}_A = \overrightarrow{\mathrm{OA}} = \begin{pmatrix}a_x\\a_y\\a_z\end{pmatrix}$: Vector from the origin to point A.

▶ **Addition, subtraction:**

$$\vec{r}_A \pm \vec{r}_B = \begin{pmatrix}a_x\\a_y\\a_z\end{pmatrix} \pm \begin{pmatrix}b_x\\b_y\\b_z\end{pmatrix} = \begin{pmatrix}a_x \pm b_x\\a_y \pm b_y\\a_z \pm b_z\end{pmatrix}$$

Vector difference $\overrightarrow{\mathrm{AB}}$: Position vector to the **final point minus** position vector to the **initial point:**

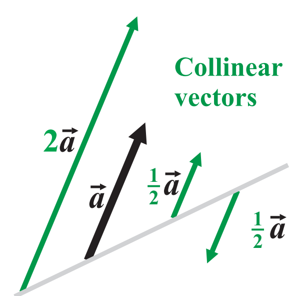
$$\overrightarrow{\mathrm{AB}} = \vec{r}_B - \vec{r}_A$$



▶ **Multiplication by a scalar (number):**

Collinear vectors $\vec{a}$ and $\vec{b}$:

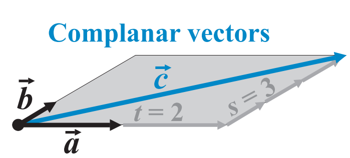
$$\vec{b} = k \cdot \vec{a} = k \cdot \begin{pmatrix}a_x\\a_y\\a_z\end{pmatrix} = \begin{pmatrix}k \cdot a_x\\k \cdot a_y\\k \cdot a_z\end{pmatrix}$$



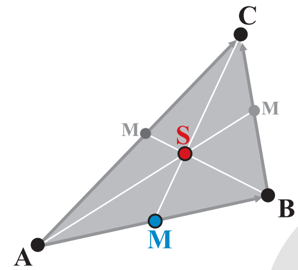
Complanar vectors: $\vec{c}$ is complanar to $\vec{a}$ and $\vec{b}$

if $\vec{c}$ can be written as a linear combination of $\vec{a}$ and $\vec{b}$:

there are $t,\ s \in \mathbb{R}$ such that $\vec{c} = t \cdot \vec{a} + s \cdot \vec{b}$ holds.



- **Midpoint M of A and B:** $\vec{r}_M = \frac{1}{2}\,(\vec{r}_A + \vec{r}_B)$

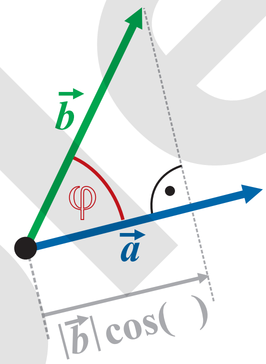


- **Centroid S of ΔABC:** $\vec{r}_S = \frac{1}{3}\,(\vec{r}_A + \vec{r}_B + \vec{r}_C)$

  center of mass, centroid see p. 7.

- **Scalar product (dot product):**

$$\vec{a}\cdot\vec{b} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} \cdot \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} = a_x\,b_x + a_y\,b_y + a_z\,b_z = |\,\vec{a}\,|\cdot|\,\vec{b}\,|\cdot\cos(\varphi)$$

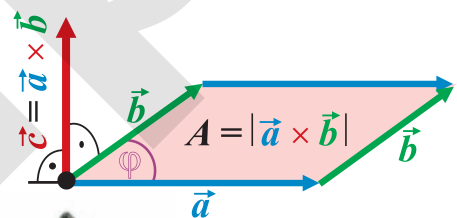


- **Angle φ between $\vec{a}$ and $\vec{b}$:** $\cos(\varphi) = \dfrac{\vec{a}\cdot\vec{b}}{|\,\vec{a}\,|\cdot|\,\vec{b}\,|}$

- **Perpendicular vectors:** $\vec{a}\perp\vec{b} \iff \vec{a}\cdot\vec{b} = 0$ if $\vec{a} \neq \vec{0}$, $\vec{b} \neq \vec{0}$.

- **Vector product (cross product):**

$$\vec{c} = \vec{a}\times\vec{b} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} \times \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} = \begin{pmatrix} a_y\,b_z - a_z\,b_y \\ a_z\,b_x - a_x\,b_z \\ a_x\,b_y - a_y\,b_x \end{pmatrix}$$



$\vec{c}\perp\vec{a}$ and $\vec{c}\perp\vec{b}$

$|\,\vec{c}\,| = |\,\vec{a}\times\vec{b}\,| = |\,\vec{a}\,|\cdot|\,\vec{b}\,|\cdot\sin(\varphi)$

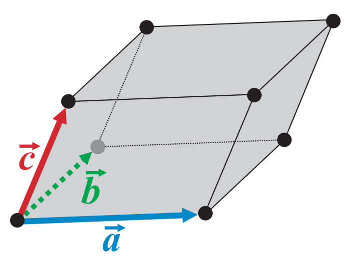
$|\,\vec{c}\,|$: Area of the parallelogram defined by $\vec{a}$ and $\vec{b}$.

Screw $\vec{c}$: Rotate $\vec{a}$ towards $\vec{b}$.

- **Triple product:**

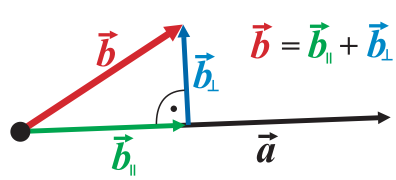
$V = |(\vec{a}\times\vec{b})\cdot\vec{c}\,| = |(\vec{b}\times\vec{c})\cdot\vec{a}\,| = |(\vec{c}\times\vec{a})\cdot\vec{b}\,|$

$|V|$: Volume of the parallelepiped defined by the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$.



- **Unit vector in the direction of $\vec{a}$:** $\vec{e}_a = \dfrac{\vec{a}}{|\,\vec{a}\,|}$

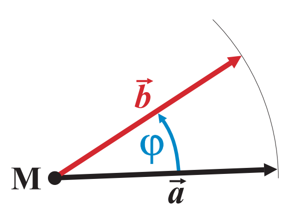
- **Decomposition of $\vec{b}$ into vectorial components parallel and perpendicular to $\vec{a}$:**



$\vec{b}_{\|} = \dfrac{(\vec{b}\cdot\vec{a})}{|\,\vec{a}\,|^2}\cdot\vec{a}$ $\qquad$ $\vec{b}_{\perp} = \vec{b} - \dfrac{(\vec{b}\cdot\vec{a})}{|\,\vec{a}\,|^2}\cdot\vec{a}$

- **Rotation of a twodimensional vector $\vec{a} = \begin{pmatrix} x \\ y \end{pmatrix}$:**

$$\vec{b} = \begin{pmatrix} x\,\cos(\varphi) - y\,\sin(\varphi) \\ x\,\sin(\varphi) + y\,\cos(\varphi) \end{pmatrix}$$



12.1 Lines (see p. 16)

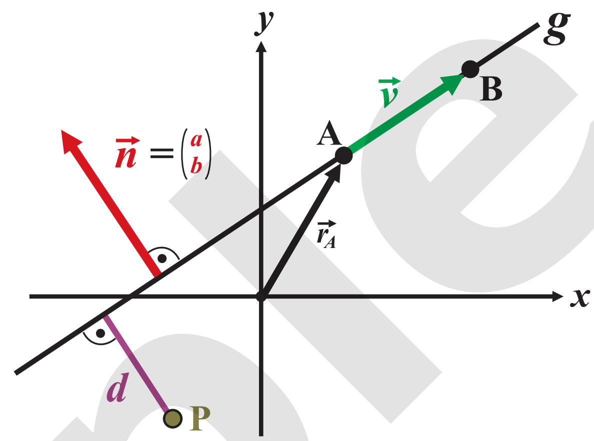
▶ Cartesian form: $g: \; a \cdot x + b \cdot y + c = 0$

- Normal vector: $\vec{n} = \begin{pmatrix} a \\ b \end{pmatrix} \perp g$
- Parallel: $g_1 \parallel g_2 \;\Leftrightarrow\; \vec{n}_1 = k \cdot \vec{n}_2$
- Perpendicular: $g_1 \perp g_2 \;\Leftrightarrow\; \vec{n}_1 \cdot \vec{n}_2 = 0$
- Angle of intersection g_1, g_2:

$$\cos(\varphi) = \frac{|\vec{n}_1 \cdot \vec{n}_2|}{|\vec{n}_1| \cdot |\vec{n}_2|}$$

- Distance of point $P(x_P \,/\, y_P)$ to g:

$$d(P, g) = \frac{|a \cdot x_P + b \cdot y_P + c|}{\sqrt{a^2 + b^2}}$$



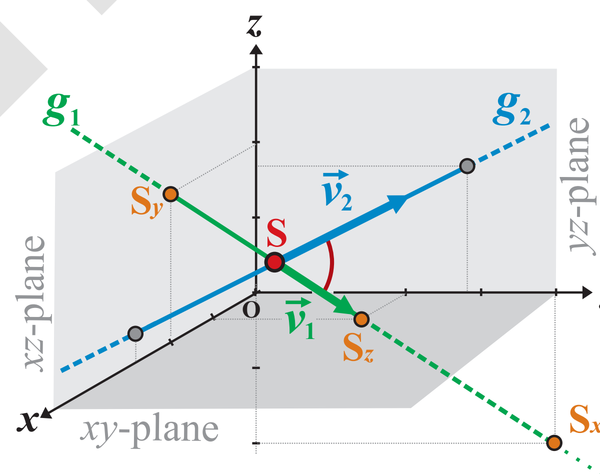
▶ Normal form, point-slope form see p. 16.

▶ Vector equation: $g: \; t \mapsto \vec{r} = \vec{r}_A + t \cdot \vec{v}$

- Direction vector $\vec{v}$: Arbitrary vector in the direction of g.
- Support point: Arbitrary point A on g.
- Parallel: $g_1 \parallel g_2 \;\Leftrightarrow\; \vec{v}_1 = k \cdot \vec{v}_2$
- Perpendicular: $g_1 \perp g_2 \;\Leftrightarrow\; \vec{v}_1 \cdot \vec{v}_2 = 0$
- Angle of intersection g_1, g_2:

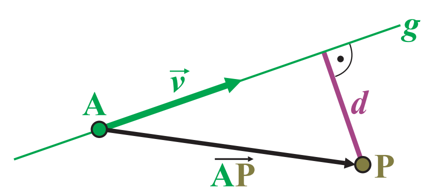
$$\cos(\varphi) = \frac{|\vec{v}_1 \cdot \vec{v}_2|}{|\vec{v}_1| \cdot |\vec{v}_2|}$$

- Track points S_x, S_y, S_z: Intersections of g with one of the main planes.



▶ Distance between point P and line

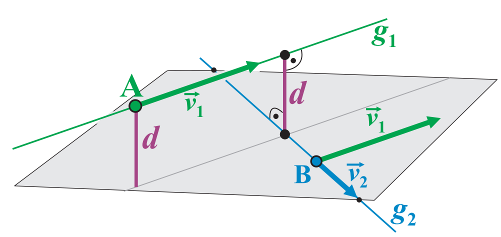
$g: \; \vec{r} = \vec{r}_A + t \cdot \vec{v}$ in space: $d(P, g) = \dfrac{|\vec{v} \times \overrightarrow{AP}|}{|\vec{v}|}$



▶ Distance of two skew lines in space

$g_1: \; \vec{r} = \vec{r}_A + t \cdot \vec{v}_1$ and $g_2: \; \vec{r} = \vec{r}_B + t \cdot \vec{v}_2$

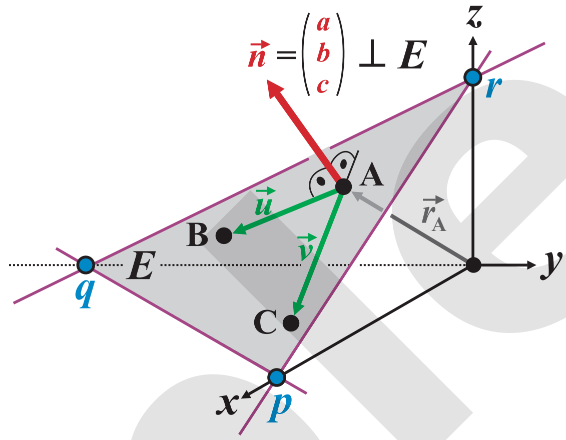
in space: $d(g_1, g_2) = \dfrac{|(\vec{v}_1 \times \vec{v}_2) \cdot (\vec{r}_B - \vec{r}_A)|}{|\vec{v}_1 \times \vec{v}_2|}$



12.2 Planes

▶ **Vector equation:** $E: \vec{r} = \vec{r}_A + t \cdot \vec{u} + s \cdot \vec{v}$

- If **3 points** A, B, C or a **support point** A (position vector $\vec{r}_A$) and **two independent directions** $\vec{u} = \overrightarrow{AB}$ and $\vec{v} = \overrightarrow{AC}$ are known.
- Each pair $t, s \in \mathbb{R}$ corresponds to exactly one point P$(x \,/\, y \,/\, z)$ with position vector $\vec{r}$ on E.



▶ **Intercept form:** $E: \frac{x}{p} + \frac{y}{q} + \frac{z}{r} = 1$ with the intercepts $\begin{cases} p, q, r \neq 0, \\ p, q, r = \infty \end{cases}$ is allowed.

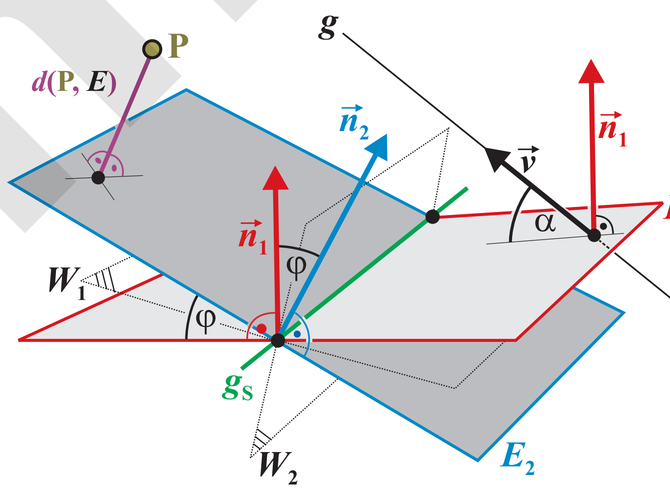
▶ **Normal form:** $E: \vec{n} \cdot (\vec{r} - \vec{r}_A) = 0$ with point $A \in E$ and $\vec{n} \perp E$.

▶ **Cartesian form:** $E: a \cdot x + b \cdot y + c \cdot z + d = 0$

- **Normal vector:**

$$\vec{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = (\vec{u} \times \vec{v}) \perp E$$

- $E_1 \parallel E_2 \iff \vec{n}_1 = k \cdot \vec{n}_2$
- $E_1 \perp E_2 \iff \vec{n}_1 \cdot \vec{n}_2 = 0$



- **Angle φ between E_1 and E_2:**

$$\cos(\varphi) = \frac{|\vec{n}_1 \cdot \vec{n}_2|}{|\vec{n}_1| \cdot |\vec{n}_2|} \qquad 0 \leq \varphi \leq 90^\circ.$$

- **Angle α between E_1 and g:**

$$\sin(\alpha) = \frac{|\vec{n}_1 \cdot \vec{v}|}{|\vec{n}_1| \cdot |\vec{v}|} \qquad 0 \leq \alpha \leq 90^\circ.$$

- **Distance P$(u \,/\, v \,/\, w)$ to E_1:**

$$d(\mathrm{P}, E) = \frac{|a \cdot u + b \cdot v + c \cdot w + d|}{\sqrt{a^2 + b^2 + c^2}}$$

- **Hesse's normal form:**

$$H(x, y, z) = \frac{a \cdot x + b \cdot y + c \cdot z + d}{\sqrt{a^2 + b^2 + c^2}} = 0$$

- **Angle bisector planes:**

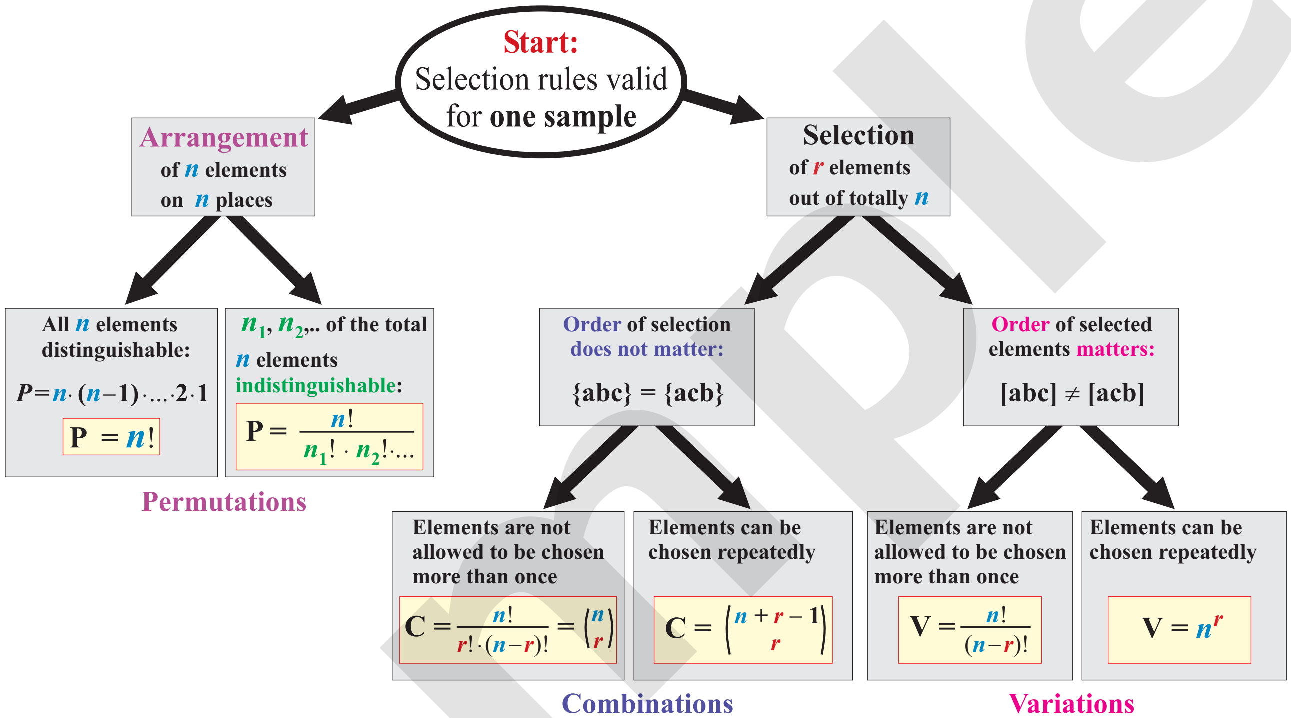
$$W_{1,2}: \; H_1(x, y, z) = \pm H_2(x, y, z)$$

▶ **Tangent plane to a sphere see p. 12.**

13 Stochastics

13.1 Combinatorics

Factorial: $n! := 1 \cdot 2 \cdot \ \ldots \ \cdot n$ $\qquad 0! := 1; \quad 1! := 1$ $\qquad$ **Binomial coefficient:** $\binom{n}{r} := \frac{n!}{r! \cdot (n-r)!}$



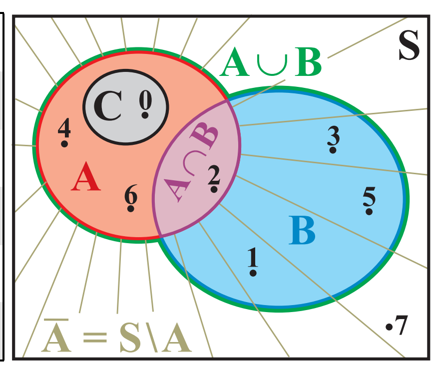
Symmetry: $\binom{n}{r} = \binom{n}{n-r}$. $\qquad$ **Recurrence relation:** $\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$.

13.2 Probability and Set Theory

- **Sample space S:** Set of all possible outcomes.
- **Events A, B, C:** Subsets of S.

Example: $\mathbf{S} = \{0, 1, 2, 3, 4, 5, 6, 7\}$, $\mathbf{A} = \{0, 2, 4, 6\}$, $\mathbf{B} = \{1, 2, 3, 5\}$. $4 \in \mathbf{A}$; $3 \notin \mathbf{A}$.

$\lvert A \rvert$	**Cardinality**	Number of elements in **A**
$A \cap B$	**Intersection**	**A and B**
$A \cup B$	**Union**	**A or B**
$\overline{A} = S \setminus A$	**Complement**	**S without A**
$C \subset A$	**Subset**	**C contained in A**
$\{\}, \emptyset$	**Empty set**	



- **Laplace-probability:** If all elements in S have the same probability to occur, then:

$$p(A) = \frac{|A|}{|S|} = \frac{\text{number of elements in A}}{\text{number of elements in S}} = \frac{\text{favorable}}{\text{possible}}$$

Impossible event $p(\emptyset) = 0$ **Certain event** $p(S) = 1$	$0 \leq p(A) \leq 1$
Complementary probability	$p(\overline{A}) = 1 - p(A)$ (Venn diagram on p. 36.)
Addition law	$p(A \cup B) = p(A) + p(B) - p(A \cap B)$
Conditional probability	$p(B \mid A)$: Probability that B occurs, under the condition that A has already occurred: „A = IF, B = THEN": $p(B \mid A) = \frac{\lvert A \cap B \rvert}{\lvert A \rvert} = \frac{p(A \cap B)}{p(A)}$ (Reduction of the sample-space from S to A.)
Multiplication law	$p(A \cap B) = p(A) \cdot p(B \mid A)$
Independent events	Events A and B are **independent** if $p(A \cap B) = p(A) \cdot p(B)$ holds.

⇒ Binomial distribution (Bernoulli) see p. 38.

13.3 Probability Distributions

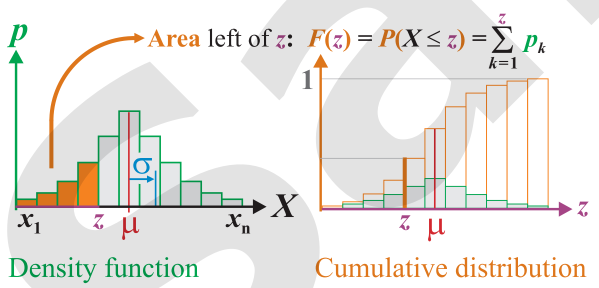
Discrete random variable:

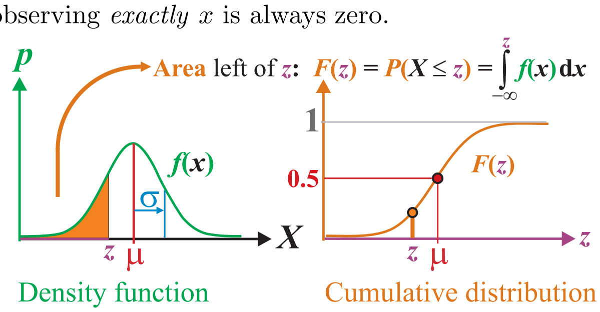
The **random variable** X takes only and *exactly* one of the n values x_1, x_2,...,x_n with the probabilities p_1, p_2,...,p_n.

Continuous random variable:

The **random variable** X may take any value $x \in \mathbb{R}$. The density function $f(x)$ evaluates the probability for *exactly* x.

Notice: For continuous variables, the probability for observing *exactly* x is always zero.





Discrete		Continuous
$p_1 + p_2 + \ldots + p_n = \sum_{k=1}^{n} p_k = 1$	**Normalization**	$\int_{-\infty}^{\infty} f(x)\, dx = 1$
$\mu = E(X) = \sum_{k=1}^{n} p_k \cdot x_k$	**Expected value (mean value)**	$\mu = E(X) = \int_{-\infty}^{\infty} f(x) \cdot x\, dx$
$\sigma^2 = \sum_{k=1}^{n} p_k \cdot (x_k - \mu)^2$	**Variance**	$\sigma^2 = \int_{-\infty}^{\infty} f(x) \cdot (x - \mu)^2\, dx$
$\sigma = \sqrt{\sigma^2} = \sqrt{\mathrm{var}(X)}$	**Standard deviation**	$\sigma = \sqrt{\sigma^2} = \sqrt{\mathrm{var}(X)}$

Let X, Y be two random variables and a, b constants. Then:

$E(a \cdot X + b \cdot Y) = a \cdot E(X) + b \cdot E(Y)$

$\mathrm{var}(a \cdot X + b) = a^2 \cdot \mathrm{var}(X)$

13.4 Binomial Distribution (discrete distribution)

The sample space of an experiment, which is repeated n times, consists of exactly two elements: S = {A, $\overline{\text{A}}$} with constant probabilities $p(A) = p$ and $p(\overline{A}) = 1 - p$. Let X be the number of times A occurs in totally n repetitions. Then:

A occurs at least once	$P(X \geq 1) = 1 - (1-p)^n$	
A occurs exactly k times	$P(X = k) = \binom{n}{k} \cdot p^{\,k} \cdot (1-p)^{n-k}$	$0 \leq k \leq n$
A occurs at most x times	$P(X \leq x) = \sum_{k=0}^{x} \binom{n}{k} \cdot p^{\,k} \cdot (1-p)^{n-k}$	$0 \leq x \leq n$
Mean value, expected value	$E(X) = n \cdot p$	
Standard deviation	$\sigma = \sqrt{n \cdot p \cdot (1-p)}$	

For $\sigma > 3$ the binomial distribution can be approximated by a normal distribution.

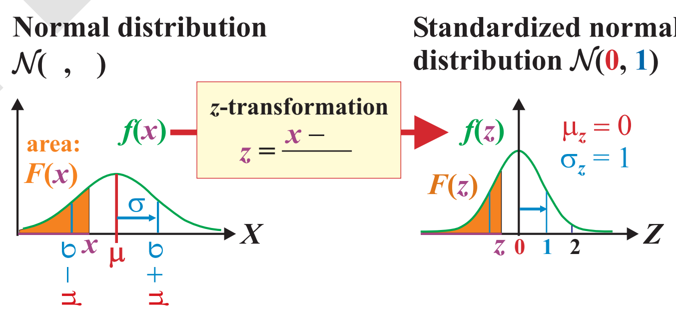
13.5 Normal Distribution (continuous distribution)

- **Density function:**

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-\frac{(x-\mu)^2}{2\,\sigma^2}} = \mathcal{N}(\mu, \sigma)$$

Standardized normal distribution:

$$f(z) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{z^2}{2}} = \mathcal{N}(0, 1)$$



Symmetry:

$f(\mu + x) = f(\mu - x)$

$f(-z) = f(+z)$

$F(-z) = 1 - F(+z)$

- **Cumulative normal distribution:**

$$F(x) = P(X \leq x) = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{x} e^{-\frac{(t-\mu)^2}{2\,\sigma^2}}\, dt$$

Probability to observe *at most* x.

Standardized normal distribution:

$$F(z) = P(Z \leq z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{t^2}{2}}\, dt$$

⇒ See table in the rear inner cover.

- **σ-environments for the normal distribution:**

$1\,\sigma$-environment	$2\,\sigma$-environment	$3\,\sigma$-environment
$p(\mid X - \mu \mid < 1\,\sigma) \approx 68.3\%$	$p(\mid X - \mu \mid < 2\,\sigma) \approx 95.4\%$	$p(\mid X - \mu \mid < 3\,\sigma) \approx 99.7\%$

13.6 Statistics: Univariate Data (one Variable)

Let $X = \{x_1,\ x_2,\ \ldots,\ x_k\}$ the values and $n_1,\ n_2,\ \ldots,\ n_k$ their **absolute frequency** of a sample of size $n = \sum\limits_{i=1}^{k} n_i = n_1 + n_2 + \ldots + n_k$. The **relative frequencies** $p(x_i) = \frac{n_i}{n}$ behave like the Laplace-probability of observing the value x_i, particularly $\sum\limits_{i=1}^{k} p(x_i) = 1$.

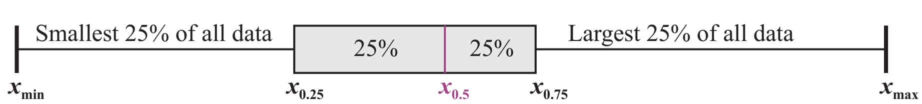
	Individual data	**Grouped data**
Data	n values $x_1,\ x_2,\ \ldots,\ x_n$	k values $x_1,\ x_2,\ \ldots,\ x_k$ with absolute frequencies $n_1,\ n_2,\ \ldots,\ n_k$
Arithmetic mean (expected value)	$\overline{x} = E(X) = \frac{1}{n}\sum\limits_{i=1}^{n} x_i$	$\overline{x} = E(X) = \frac{1}{n}\sum\limits_{i=1}^{k} n_i\, x_i = \sum\limits_{i=1}^{k} p(x_i)\cdot x_i$
Median	The median $x_{0.5}$ of the values of an ordered sample is • the value of the middle item, if n is odd. • the mean of the middle two items, if n is even.	
Mode	Value that appears most often in a set of data.	
Average linear deviation from mean	$s_m = \frac{1}{n}\sum\limits_{i=1}^{n} \lvert x_i - \overline{x}\rvert$	$s_m = \frac{1}{n}\sum\limits_{i=1}^{k} n_i\,\lvert x_i - \overline{x}\rvert$
Range	$R = x_{\max} - x_{\min}$	
Variance*	$s_x^2 = \frac{1}{n-1}\sum\limits_{i=1}^{n}(x_i - \overline{x})^2$	$s_x^2 = \frac{1}{n-1}\sum\limits_{i=1}^{k} n_i\,(x_i - \overline{x})^2$ or $s_x^2 = \sum\limits_{i=1}^{k} p(x_i)\cdot(x_i - \overline{x})^2 = E(X^2) - \big(E(X)\big)^2$

If the values $x_1,\ x_2,\ \ldots,\ x_n$ represent an entire population or if we are interested in the variation within the sample itself, the denominator is n (instead of $n-1$).

▶ **Standard deviation:** $s_x = \sqrt{s_x^2}$

▶ **Variation coefficient:** $V = \frac{s_x}{\overline{x}} \cdot 100\%$ is used to compare different samples.

▶ **Box plot:** Evaluate the median $x_{0.5}$, the upper ($x_{0.75}$) and lower ($x_{0.25}$) quartiles, the smallest ($x_{\min}$) and the largest ($x_{\max}$) sample. Graphical representation:



▶ **Inequality of Chebychev:**

For a sample with mean $\overline{x}$ and variance s_x^2, the probability p for an observation x to be found within a range of $\pm\lambda$ from the mean is given by $p(\lvert x - \overline{x}\rvert < \lambda) \ \geq\ 1 - \frac{s_x^2}{\lambda^2}$.

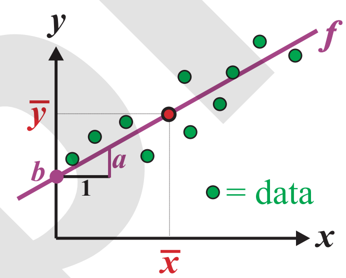
13.7 Statistics: Bivariate Data, Regression and Correlation

Let $(x_1, y_1), (x_2, y_2), \ldots (x_n, y_n)$ be n pairs of observations. To describe the dependency between x and y, a model function $y = f(x)$ which depends on the parameters a, b,... is fitted to the data such that the mean square deviation of $y_i - f(x)$ becomes a minimum:

$$F(x) = \sum_{i=1}^{n} \left(y_i - f(x_i) \right)^2 \longrightarrow \text{minimum}$$

Linear Regression:

Model function: $y = f(x) = a \cdot x + b$ with



- **Slope:** $$a = \frac{\sum_{i=1}^{n}(x_i - \overline{x})(y_i - \overline{y})}{\sum_{i=1}^{n}(x_i - \overline{x})^2} = \frac{c_{xy}}{s_x^2} = r_{xy} \cdot \frac{s_y}{s_x}$$
- **y-Intercept:** $b = \overline{y} - a \cdot \overline{x}$

Alternative: System of linear equations to calculate a and b of the linear regression:

$$\left| \begin{array}{lcl} \left(\sum_{i=1}^{n} x_i^2\right) \cdot a + \left(\sum_{i=1}^{n} x_i\right) \cdot b & = & \sum_{i=1}^{n} x_i \cdot y_i \\ \left(\sum_{i=1}^{n} x_i\right) \cdot a + n \cdot b & = & \sum_{i=1}^{n} y_i \end{array} \right|$$

Correlation coefficient:

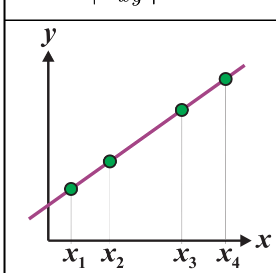
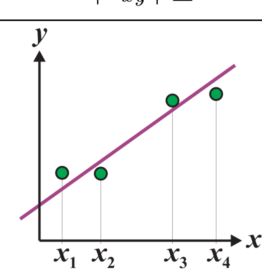
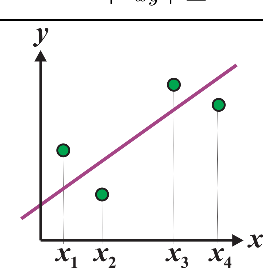
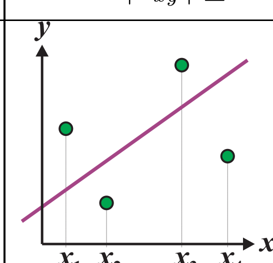
$$r_{xy} = \frac{\sum_{i=1}^{n}(x_i - \overline{x})(y_i - \overline{y})}{\sqrt{\sum_{i=1}^{n}(x_i - \overline{x})^2 \cdot \sum_{i=1}^{n}(y_i - \overline{y})^2}} = \frac{c_{xy}}{s_x \cdot s_y}$$

Covariance:

$$c_{xy} = \frac{1}{n-1} \sum_{i=1}^{n}(x_i - \overline{x})(y_i - \overline{y})$$

$$c_{xy} = E(X \cdot Y) - E(X) \cdot E(Y)$$

r_{xy} describes the strength of correlation between x and y:

Correlation coefficient r_{xy}			
full	**strong**	**medium**	**weak to none**
$\lvert r_{xy} \rvert = \mathbf{1}$	$1 > \lvert r_{xy} \rvert \geq 0.7$	$0.7 > \lvert r_{xy} \rvert \geq 0.3$	$0.3 > \lvert r_{xy} \rvert \geq \mathbf{0}$
			

14 Mathematical Symbols

$A \Rightarrow B$	**Implication:** A implies B
$A \Leftrightarrow B$	**Equivalence:** A is equivalent to B
$\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$	**Sets of numbers** (see p. 2)
$\mathbb{D}$, $\mathbb{W}$	**Domain**, **range** (see p. 14)
$f : x \mapsto y = f(x)$	y is a function of x (see p. 14)
$A = \{a, b, c, ...\}$	**Set** A consisting of the elements a, b, c,....
$[a,\ b]$	**Interval** between (**including**) a and b
$(a,\ b)$	**Interval** between (but without) a and b example: $(2,\ 5]$ = set of all x, such that $2 < x \leq 5$
$5 \in \mathbb{N}$	**Element**: 5 is element of $\mathbb{N}$, that is, 5 is a natural number
$1.5 \notin \mathbb{N}$	**Not element**: 1.5 is not in the set $\mathbb{N}$
$\mathrm{P} \in f$	Point P is on the graph of the function f
$A \subset B$	**Subset**: set A is part of B
$A \cap B$	A **Intersect** B: elements that are in A **and** in B
$g \cap E$	Line g **intersected** with plane E
$A \cup B$	A **union** B: elements that are in A **or** in B or in both.
$A \setminus B$	A **without** B: elements that are in A but **not** in B
$\mid$	**Condition (if)**. Examples: $\mathbb{D} = \{x \in \mathbb{R} \mid x \leq 1\}$ = set of all x smaller or equal 1 $p(B \mid A)$ = Probability of B to occur, **if** A already ocurred
$\forall$	**For all**: $\forall\, x \in \mathbb{R}$...
$\exists$	**There is**: $\exists\, x \in \mathbb{R}$ there is a real number x such that...

The Greek Alphabet

A	α	Alpha	H	η	Eta	N	ν	Nu	T	τ	Tau
B	β	Beta	Θ	θ, ϑ	Theta	Ξ	ξ	Xi	Y	υ	Upsilon
Γ	γ	Gamma	I	ι	Iota	O	o	Omicron	Φ	ϕ, φ	Phi
Δ	δ	Delta	K	κ	Kappa	Π	π	Pi	X	χ	Chi
E	ϵ, ε	Epsilon	Λ	λ	Lambda	P	ρ	Rho	Ψ	ψ	Psi
Z	ζ	Zeta	M	μ	Mu	Σ	σ, ς	Sigma	Ω	ω	Omega

Index