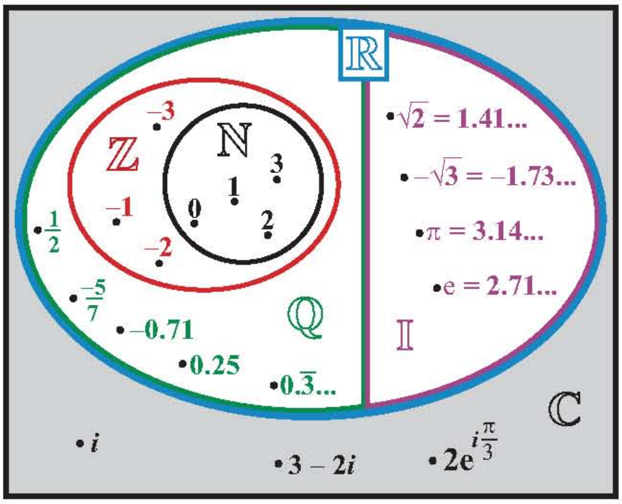
Mathematics Formulary

To get a register, please cut out the grey parts on the margins

1 Number Systems



- **Natural numbers:** $\mathbb{N} = \{0, 1, 2, 3, ...\}$.
- **Integers:** $\mathbb{Z} = \{0, \pm 1, \pm 2, ...\}$.
- **Rational numbers:** set of all fractions: $\mathbb{Q} = \{\frac{m}{n} \mid m, n \in \mathbb{Z},\ n \neq 0\}$. Numbers with periodic or terminating decimal expansion.
- **Irrational numbers** $\mathbb{I}$: numbers with infinite nonperiodic decimal expansion.
- **Real numbers** $\mathbb{R}$: union of rational and irrational numbers.
- **Complex numbers:** $\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}\}$ with $i^2 = -1$.

Complex Numbers

▶ **Imaginary unit:** $i^2 = -1$

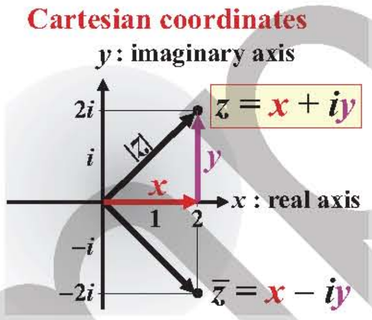
▶ **Euler's formula:**

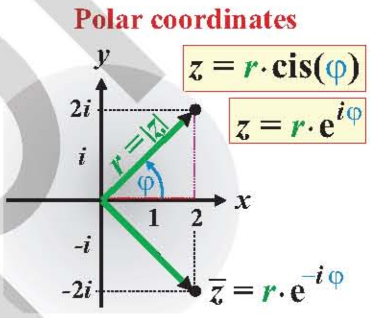
$$e^{i\varphi} = \cos(\varphi) + i\,\sin(\varphi)$$

$e^{i\varphi} = \text{cis}(\varphi), \quad |e^{i\varphi}| = 1$

▶ **Argand diagram:**

xy-plane of the complex numbers.





Complex number	z	$z = x + i\,y$ $\begin{cases} x: \text{real part} \\ y: \text{imaginary part} \end{cases}$	$z = r \cdot e^{i\varphi} = r \cdot \text{cis}(\varphi)$
Complex conjugate	$\overline{z}$	$\overline{z} = x - i\,y$	$\overline{z} = r \cdot e^{-i\varphi}$
Modulus **Angle**	$\lvert z \rvert$ φ	$\lvert z \rvert = \sqrt{z \cdot \overline{z}} = \sqrt{x^2 + y^2}$ $x = r \cdot \cos(\varphi)$ $y = r \cdot \sin(\varphi)$	$\lvert z \rvert = r = \sqrt{x^2 + y^2}$ $\tan(\varphi) = \frac{y}{x}$ $\varphi = \arg(z)$
Addition **Subtraction**	$z_1 + z_2$ $z_1 - z_2$	$(x_1 \pm x_2) + i\,(y_1 \pm y_2)$	
Multiplication **Division** $(z_2 \neq 0)$ **Inverse** $(z \neq 0)$	$z_1 \cdot z_2$ $\frac{z_1}{z_2}$ $\frac{1}{z}$	$(x_1\,x_2 - y_1\,y_2) + i\,(x_1\,y_2 + x_2\,y_1)$ $\frac{z_1 \cdot \overline{z_2}}{\lvert z_2 \rvert^2} = \frac{(x_1\,x_2 + y_1\,y_2) + i\,(x_2\,y_1 - x_1\,y_2)}{{x_2}^2 + {y_2}^2}$ $\frac{\overline{z}}{\lvert z \rvert^2} = \frac{x - i\,y}{x^2 + y^2}$	$r_1 \cdot r_2 \cdot \text{cis}(\varphi_1 + \varphi_2)$ $\frac{r_1}{r_2} \cdot \text{cis}(\varphi_1 - \varphi_2)$ $\frac{1}{r} \cdot \text{cis}(-\varphi)$
Powers **Roots**	z^n $\sqrt[n]{z}$	$r^n \cdot (\cos(n\,\varphi) + i\,\sin(n\,\varphi)) = r^n \cdot e^{in\varphi}$ $\sqrt[n]{r} \cdot \left(\cos\left(\frac{\varphi + 2\pi k}{n}\right) + i\,\sin\left(\frac{\varphi + 2\pi k}{n}\right)\right), \quad k = 0, 1, \ldots, (n-1)$	

2 Algebra

2.1 Addition and Multiplication, Basic Laws

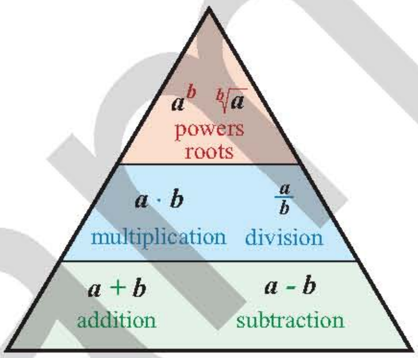
	Addition	Multiplication
Commutative law	$a+b=b+a$	$a\cdot b=b\cdot a$
Associative law	$(a+b)+c=a+(b+c)=a+b+c$	$(a\cdot b)\cdot c=a\cdot(b\cdot c)=a\cdot b\cdot c$
Distributive law	$a\cdot(b\pm c)=a\cdot b\pm a\cdot c$	
Neutral element	$a+0=0+a=a$	$a\cdot 1=1\cdot a=a$
Inverse element	$a+(-a)=(-a)+a=0$	$a\cdot(a^{-1})=(a^{-1})\cdot a=1$

2.2 Order of Operators

Brackets → Exponents (and roots) → Division → Multiplication → Addition → Subtraction

Optional brackets:

- $-1^2=-(1)^2=-1$
- $2\cdot 3^4=2\cdot(3^4)=162$
- $4/2+3=(4/2)+3=5$
- $2+3\cdot 4=2+(3\cdot 4)=14$



Mandatory brackets:

- $(-1)^2=(-1)\cdot(-1)=+1$
- $(2\cdot 3)^4=6^4=1296$
- $4/(2+3)=4/5=0.8$
- $(2+3)\cdot 4=5\cdot 4=20$

2.3 Equivalence Transformations

The following transformations leave the set of solutions unchanged:

Equation $a=b$		Inequality $a<b$
$a\pm c=b\pm c$	**Addition, subtraction**	$a\pm c<b\pm c$
$a\cdot c=b\cdot c$	**Multiplication** by $c\neq 0$	$a\cdot c<b\cdot c$ if $c>0$ $a\cdot c>b\cdot c$ if $c<0$
$\frac{a}{c}=\frac{b}{c}$	**Division** by $c\neq 0$	$\frac{a}{c}<\frac{b}{c}$ if $c>0$ $\frac{a}{c}>\frac{b}{c}$ if $c<0$
$\frac{1}{a}=\frac{1}{b}$	**Reciprocal** $(a,b\neq 0)$	$\frac{1}{a}<\frac{1}{b}$ if $a\cdot b<0$ $\frac{1}{a}>\frac{1}{b}$ if $a\cdot b>0$

2.4 Binomial Formulae, Binomial Theorem

Binomial formulae:

1st **formula:** $(a+b)^2 = a^2 + 2\cdot a\cdot b + b^2$

2nd **formula:** $(a-b)^2 = a^2 - 2\cdot a\cdot b + b^2$

3rd **formula:** $(a+b)\cdot(a-b) = a^2 - b^2$

- $a^2 + b^2$ irreducible in $\mathbb{R}$.
- $a^3 + b^3 = (a+b)\cdot(a^2 - a\cdot b + b^2)$
- $a^3 - b^3 = (a-b)\cdot(a^2 + a\cdot b + b^2)$
- $a^n - b^n = (a-b)\cdot\sum_{k=0}^{n-1} a^{n-1-k}\cdot b^k$

Binomial theorem:

$$(a+b)^n = \underbrace{\binom{n}{0}}_{1} a^n\, b^0 + \binom{n}{1} a^{n-1}\, b^1 + \binom{n}{2} a^{n-2}\, b^2 + \ldots + \underbrace{\binom{n}{n}}_{1} a^0\, b^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k}\cdot b^k$$

with the binomial coefficients $\binom{n}{k} = \frac{n!}{k!\cdot(n-k)!}$.

Factorial: $n! = 1\cdot 2\cdot\ \ldots\ \cdot n$, $\quad 0! = 1! = 1$. (see combinatorics on p. 30).

For $(a-b)^n$ the sign is *alternating*: $(a-b)^3 = +a^3 - 3\,a^2 b + 3\,a\,b^2 - b^3$.

Pascal's triangle and binomial theorem:

1	$n = 0$	$\binom{0}{0}$	$(a+b)^0 = 1$
1 + 1	$n = 1$	$\binom{1}{0}$ $\binom{1}{1}$	$(a+b)^1 = 1a^1 + 1b^1$
1 + 2 + 1	$n = 2$	$\binom{2}{0}$ $\binom{2}{1}$ $\binom{2}{2}$	$(a+b)^2 = 1a^2 + 2a^1b^1 + 1b^2$
1 + 3 + 3 + 1	$n = 3$	$\binom{3}{0}$ $\binom{3}{1}$ $\binom{3}{2}$ $\binom{3}{3}$	$(a+b)^3 = 1a^3 + 3a^2b^1 + 3a^1b^2 + 1b^3$
1 4 6 4 1	$n = 4$	$\binom{4}{0}$ $\binom{4}{1}$ $\binom{4}{2}$ $\binom{4}{3}$ $\binom{4}{4}$	$(a+b)^4 = 1a^4 + 4a^3b^1 + 6a^2b^2 + 4a^1b^3 + 1b^4$

2.5 Fractions

Addition Subtraction	$\frac{a}{b} \pm \frac{x}{y} = \frac{a\cdot y}{b\cdot y} \pm \frac{x\cdot b}{y\cdot b} = \frac{a\cdot y \pm x\cdot b}{b\cdot y}$ $\quad b,\ y \neq 0$	► put onto the common denominator, then add the numerators.
Multiplication	$\frac{a}{b}\cdot\frac{x}{y} = \frac{a\cdot x}{b\cdot y}$ $\quad b,\ y \neq 0$	► multiply numerators and denominators with each other.
Division	$\frac{a}{b} : \frac{x}{y} = \frac{\frac{a}{b}}{\frac{x}{y}} = \frac{a}{b}\cdot\frac{y}{x}$ $\quad b,\ x,\ y \neq 0$	► dividing by a fraction: Multiplying by its reciprocal.

2.6 Powers

Definition: $a^n = \underbrace{a \cdot a \cdot \ldots \cdot a}_{n \text{ factors}}$ is called the n^{th} power of a, where $\begin{cases} a \in \mathbb{R} & : \text{base} \\ n \in \mathbb{N} & : \text{exponent.} \end{cases}$

Particularly: $a^1 = a$ and $\begin{cases} a^0 = 1, & \text{if } a \neq 0 \\ 0^n = 0, & \text{if } n > 0. \end{cases}$

- **Negative exponents** $\Rightarrow$ **denominators:** $a^{-n} = \frac{1}{a^n}$ $\quad a \neq 0.$
- **Rational exponents** $\Rightarrow$ **roots:** $a^{\frac{m}{n}} = \sqrt[n]{a^m}$ $\quad a \geq 0, \quad n > 0.$

particularly: $a^{\frac{1}{n}} = \sqrt[n]{a}$ $\quad$ square root: $n = 2$: $\sqrt{a} = a^{\frac{1}{2}}$

Power laws

Same base	$a^n \cdot a^m = a^{n+m}$	$\frac{a^n}{a^m} = a^{n-m}$	$a \neq 0$
Same exponent	$a^n \cdot b^n = (ab)^n$	$\frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n$	$b \neq 0$
Powers of powers	$(a^n)^m = a^{n \cdot m} = (a^m)^n$		

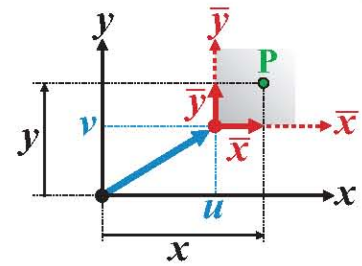
2.7 Logarithms (see also p. 16)

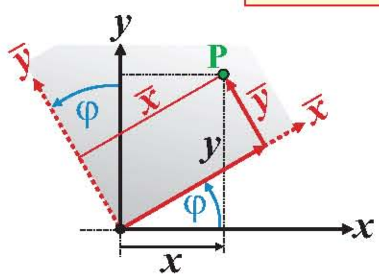
Definition	$\log_a(x) = y \quad \Leftrightarrow \quad a^y = x$	$a,\ x > 0; \quad a \neq 1$	
Multiplication Division	$\log_a(x \cdot y) = \log_a(x) + \log_a(y)$	$\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$	
Powers	$\log_a(x^y) = y \cdot \log_a(x)$	$x > 0$	
Change of base	$\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$	$a > 0;\ a \neq 1$ $b > 0;\ b \neq 1$	especially: $\log_a(x) = \frac{\ln(x)}{\ln(a)}$

2.8 Translation, Rotation of the Coordinate System

Translation of $\begin{pmatrix} u \\ v \end{pmatrix}$: $\quad \begin{aligned} \overline{x} &= x - u \\ \overline{y} &= y - v \end{aligned}$

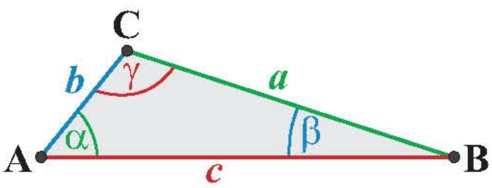
Rotation by φ : $\quad \begin{aligned} \overline{x} &= x\cos(\varphi) + y\sin(\varphi) \\ \overline{y} &= -x\sin(\varphi) + y\cos(\varphi) \end{aligned}$



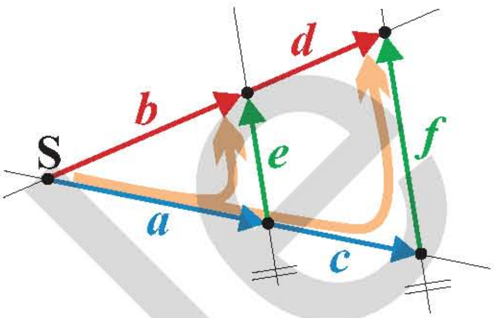


3 Plane Geometry

3.1 Triangles



- **Sum of angles:** $\alpha + \beta + \gamma = 180°$
- **Triangle inequality:** $c < a + b$
- **Intercept theorems (proportionality):**
  Two triangles are called similar if they have the **same angles**. Equivalently, the **ratios** of their sides are **equal**.



  1st Intercept theorem: $\frac{a}{b} = \frac{c}{d} = \frac{a+c}{b+d}$

  2nd Intercept theorem: $\frac{a}{e} = \frac{a+c}{f}$

- **Sine rule:** $\frac{a}{\sin(\alpha)} = \frac{b}{\sin(\beta)} = \frac{c}{\sin(\gamma)} = 2R$ with R : radius of the circumcircle.

- **Cosine rule:** $c^2 = a^2 + b^2 - 2 \cdot a \cdot b \cdot \cos(\gamma)$ and cyclic permutations: 

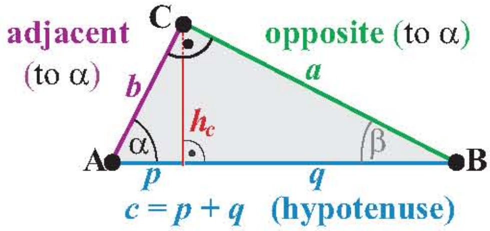
- **Area:** $A_\Delta = \frac{1}{2}\,(\text{base} \cdot \text{height}) = \frac{c \cdot h_c}{2} = \frac{b \cdot h_b}{2} = \frac{a \cdot h_a}{2}$

  ► **two sides and their enclosed angle:** $A_\Delta = \frac{b \cdot c}{2} \cdot \sin(\alpha)$ and cyclic: 

  ► **three sides (Heron):** $A_\Delta = \sqrt{s(s-a)(s-b)(s-c)}$ with the semi-perimeter $s = \frac{1}{2}(a+b+c)$

  ► **three angles and *R*:** $A_\Delta = 2\,R^2 \cdot \sin(\alpha) \cdot \sin(\beta) \cdot \sin(\gamma)$ R : Radius of the circumcircle.

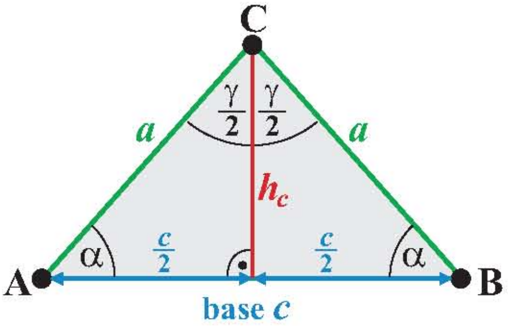
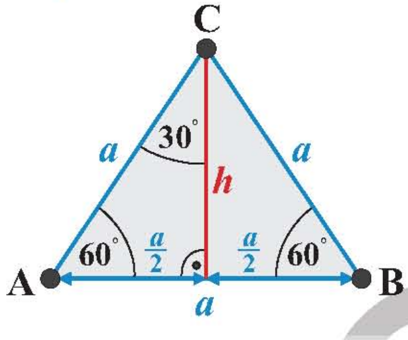
3.2 Right-angled Triangle



- **Pythagoras' theorem:** $c^2 = a^2 + b^2$
- **Altitude theorem:** ${h_c}^2 = p \cdot q$
- **Euclid's theorem:**
  $a^2 = c \cdot q$ resp. $b^2 = c \cdot p$
- **Trigonometric functions:** (see p. 17)
  $\sin(\alpha) = \frac{a}{c}$ $\cos(\alpha) = \frac{b}{c}$ $\tan(\alpha) = \frac{a}{b}$

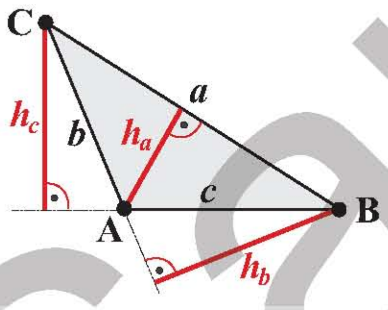
  mnemonic:
  SOH CAH TOA

3.3 Isosceles and Equilateral Triangles

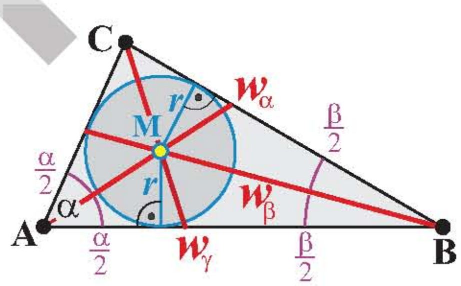
Isosceles triangle	Equilateral triangle
	
► h_c bisects base c. ► h_c bisects angle γ.	► height: $h = \frac{\sqrt{3}}{2}\,a$ ► area: $A = \frac{\sqrt{3}}{4}\,a^2$ ► radius circumcircle: $R = \frac{\sqrt{3}}{3}\,a = \frac{2}{3}\,h$ ► radius incircle: $r = \frac{\sqrt{3}}{6}\,a = \frac{1}{3}\,h$

3.4 Lines in a Triangle

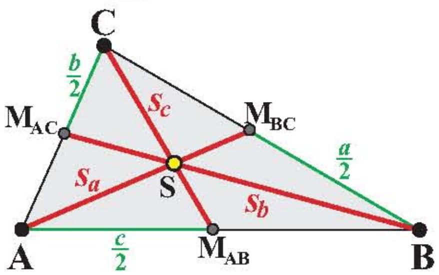
Heights are straight lines through a vertex **perpendicular** to the opposite side.



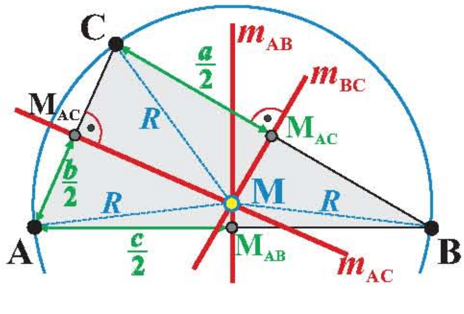
Angle bisectors bisect an angle of the triangle. Each point on an angle bisector has the **same distance** from the adjacent sides. Angle bisectors intersect at the **center M of the incircle**.



Medians are the lines from a vertex to the midpoint of the opposite side. They intersect in the **ratio 2:1**. The point of coincidence is the **centroid S** (center-of-mass) of the triangle. See also p. 27.

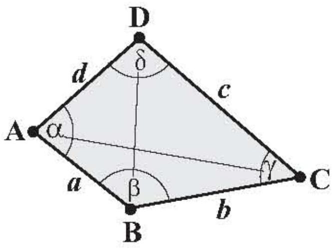


Perpendicular bisectors are the set of points having the **same distance** from two vertices of the triangle. They intersect at the **center M of the circumcircle**.



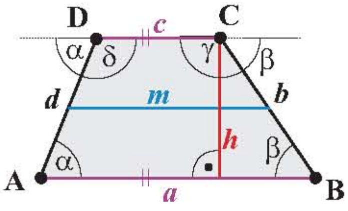
3.5 Quadrilaterals

Quadrilateral



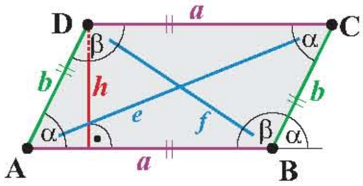
► $\alpha + \beta + \gamma + \delta = 360°$

Trapezium, trapezoid



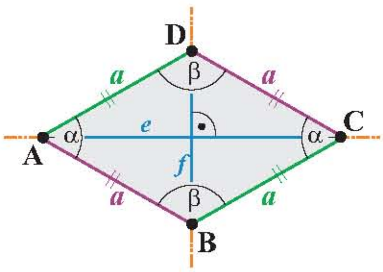
► $A = \frac{a+c}{2} \cdot h = m \cdot h$

Parallelogram, rhomboid



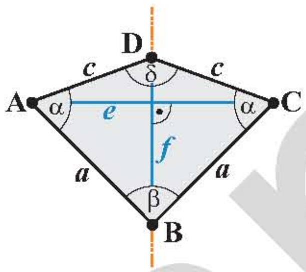
► $A = a \cdot h = a \cdot b \cdot \sin(\alpha)$

Rhombus



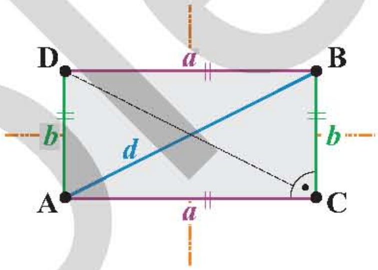
► $A = \frac{e \cdot f}{2} = a^2 \cdot \sin(\alpha)$

Kite



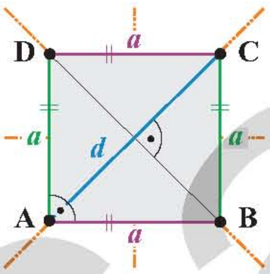
► $A = \frac{e \cdot f}{2} = a \cdot c \cdot \sin(\alpha)$

Rectangle



► $A = a \cdot b$

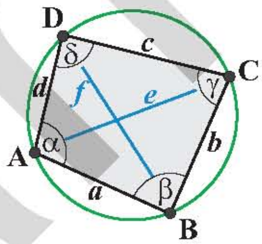
Square



► $A = a^2$

► $d = a \cdot \sqrt{2}$

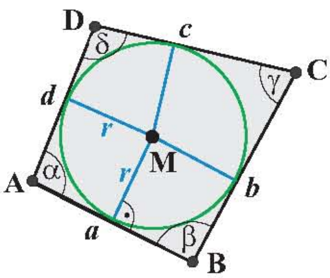
Cyclic quadrilateral



► $\alpha + \gamma = \beta + \delta = 180°$

► $a \cdot c + b \cdot d = e \cdot f$

Tangent quadrilateral

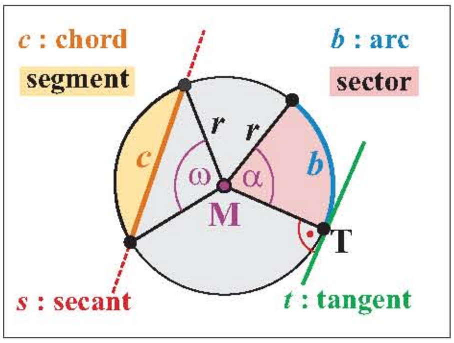


► $a + c = b + d$

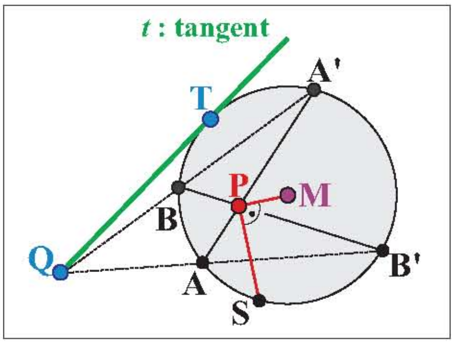
► $A = r \cdot \frac{a+b+c+d}{2}$

Axis of symmetry are shown in orange color.

3.6 Circle



Circumference	$U = 2\pi \cdot r$
Arc length	$b = 2\pi r \cdot \frac{\alpha}{360°}$
Area	$A = \pi \cdot r^2$
Sector	$A_{\text{Sec}} = \pi r^2 \cdot \frac{\alpha}{360°} = \frac{b\,r}{2}$



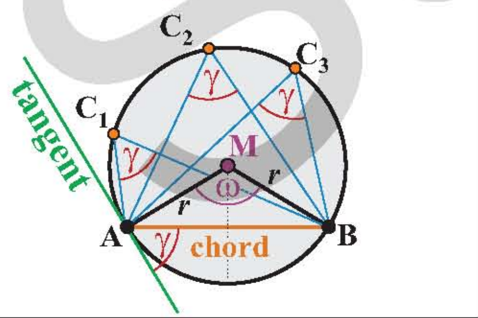
Segment	$A_{\text{Seg}} = \pi\, r^2\, \frac{\omega}{360°} - \frac{1}{2}\, r^2 \,\sin(\omega)$
Intersecting chord theorem	$\overline{PA} \cdot \overline{PA'} = \overline{PB} \cdot \overline{PB'} = \overline{PS}^2$
Intersecting secant theorem	$\overline{QB} \cdot \overline{QA'} = \overline{QA} \cdot \overline{QB'} = \overline{QT}^2$

Equation of circle C with center $M(u\ /\ v)$ and radius r:

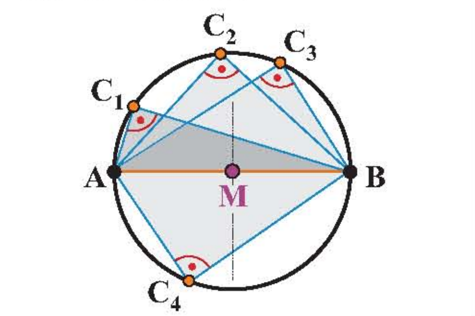
- Center form: $C:\ (x-u)^2 + (y-v)^2 = r^2$
- Expanded form: $C:\ x^2 + y^2 + a \cdot x + b \cdot y + c = 0$
- Tangent t to C at point $T(x_0\ /\ y_0)$: $t:\ (x-u) \cdot (x_0 - u) + (y-v) \cdot (y_0 - v) = r^2$

Circle Angle Theorems

Central angle theorem:



Thales' theorem:

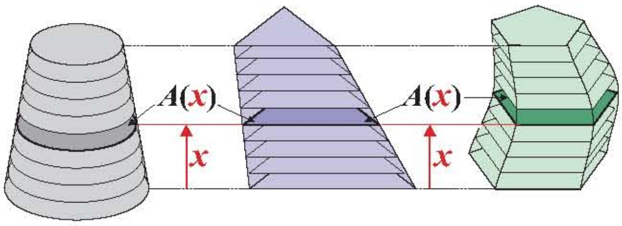


- ► equal inscribed angle γ.
- ► central angle $\omega = 2 \cdot \gamma$.
- ► inscribed angle $\gamma = 90°$.

4 Stereometry

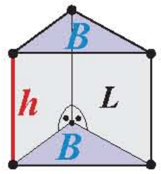
4.1 Cavalieri's Principle

Two solids have the same volume if their cross-sections $A(x)$ have the same area at all levels x.

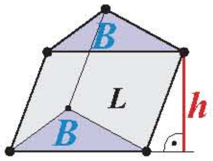


4.2 Prisms and Cylinders (Congruent, Parallel Base and Top Face B)

Right prism

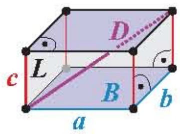


Oblique prism



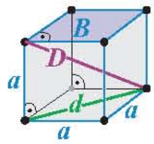
- B : Base area, L : Lateral area.
- h : Height.
- Volume: $V = B \cdot h$
- Surface area: $A = 2 \cdot B + L$

Cuboid



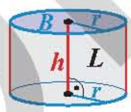
- $V = a \cdot b \cdot c$
- $A = 2(a \cdot b + a \cdot c + b \cdot c)$
- $D = \sqrt{a^2 + b^2 + c^2}$

Cube



- $V = a^3$
- $A = 6 \cdot a^2$
- $D = a \cdot \sqrt{3}, \quad d = a \cdot \sqrt{2}$

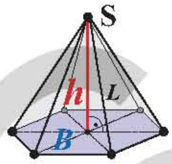
Cylinder



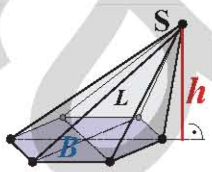
- $V = \pi r^2 \cdot h$
- $A = 2 \cdot \pi r^2 + 2\pi r \cdot h$
- $L = 2\pi r \cdot h$

4.3 Pyramids and Cones

Right pyramid

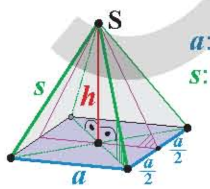


Oblique pyramid



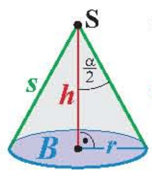
- B : Base area, L : Lateral area.
- h : Height.
- Volume: $V = \frac{1}{3} \cdot B \cdot h$
- Surface area: $A = B + L$

Right, square pyramid



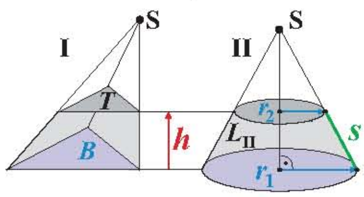
- $V = \frac{1}{3}\, a^2 \cdot h$
- $A = a^2 + L$
- $s = \sqrt{h^2 + \frac{a^2}{2}}$

Right circular cone



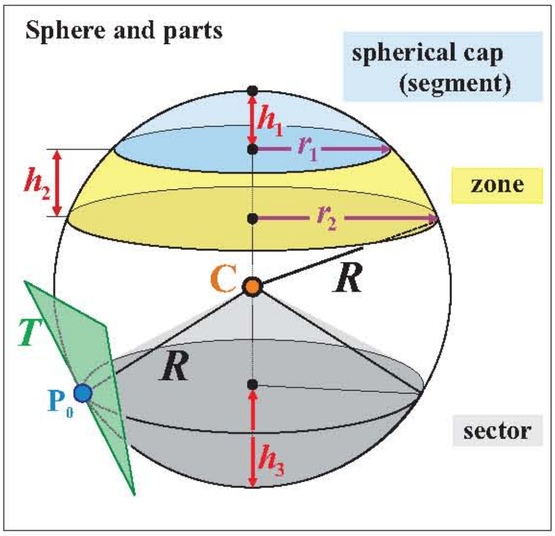
- $V = \frac{1}{3}\, \pi r^2 \cdot h$
- $A = \pi r^2 + \pi r s, \quad L = \pi r s$
- $s = \sqrt{h^2 + r^2}$

Frustum of { pyramid / cone



- $V_{\text{I}} = \frac{h}{3}\,(B + \sqrt{BT} + T)$
- $V_{\text{II}} = \frac{\pi h}{3}\,({r_1}^2 + r_1 r_2 + {r_2}^2)$
- $L_{\text{II}} = \pi \cdot s \cdot (r_1 + r_2)$

4.4 Sphere



Volume: $V = \frac{4}{3}\pi \cdot R^3$

- ► cap: $V = \frac{1}{3}\pi \cdot {h_1}^2 \cdot (3R - h_1)$
- ► zone: $V = \frac{1}{6}\pi \cdot h_2 \cdot (3{r_1}^2 + 3{r_2}^2 + {h_2}^2)$
- ► sector: $V = \frac{2}{3}\pi \cdot R^2 \cdot h_3$

Surface area: $A = 4\pi \cdot R^2$

- ► cap: $L = 2\pi R \cdot h_1$ (lateral area)
- ► zone: $L = 2\pi R \cdot h_2$ (lateral area)
- ► sector: $A = 2\pi R \cdot h_3 + \pi R\sqrt{2Rh_3 - {h_3}^2}$

Equation of a sphere S with center $C(u / v / w)$ and radius R:

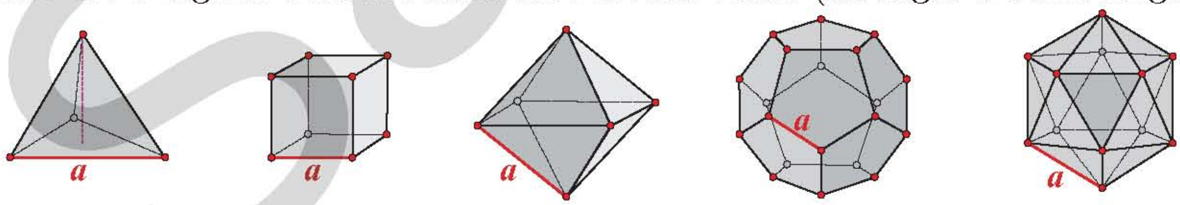
- **Center form:** $S: (x-u)^2 + (y-v)^2 + (z-w)^2 = R^2$
- **Expanded form:** $S: x^2 + y^2 + z^2 + a \cdot x + b \cdot y + c \cdot z + d = 0$
- **Tangent plane** T to a sphere S at a point $P_0(x_0 / y_0 / z_0)$:

$T: (x-u)\cdot(x_0-u) + (y-v)\cdot(y_0-v) + (z-w)\cdot(z_0-w) = R^2$ (see p. 29)

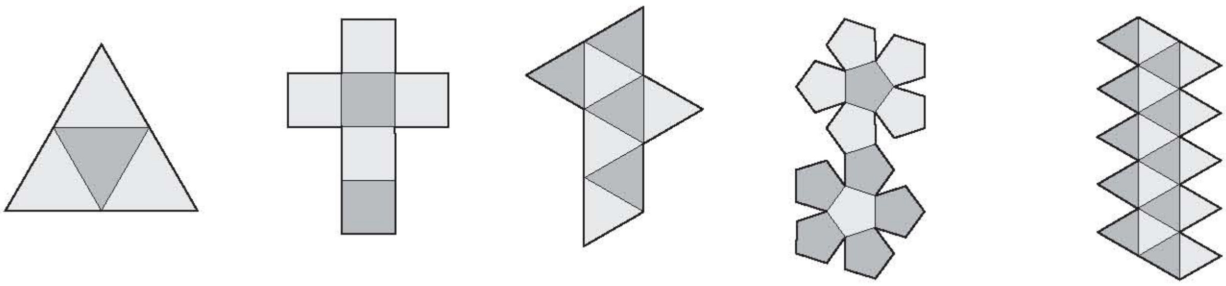
4.5 Polyhedra, Platonic Solids

Euler's Polyhedron Theorem: $V + F = E + 2$ with:
- V : number of vertices
- F : number of faces
- E : number of edges.

There are 5 regular convex solids: the Platonic solids (all edges of same length)



***Tetra*hedron** (4 faces)	***Hexa*hedron** (6 faces)	***Octa*hedron** (8 faces)	***Dodeca*hedron** (12 faces)	***Icosa*hedron** (20 faces)

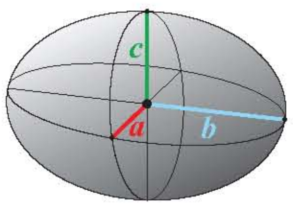


Stereometry

	Volume V	Surface A	Radius R of the circumsphere	Radius r of the insphere
Tetra-hedron	$\frac{\sqrt{2}}{12}\,a^3$	$\sqrt{3}\,a^2$	$\frac{\sqrt{6}}{4}\,a$	$\frac{\sqrt{6}}{12}\,a$
Hexa-hedron	a^3	$6\,a^2$	$\frac{\sqrt{3}}{2}\,a$	$\frac{1}{2}\,a$
Octa-hedron	$\frac{\sqrt{2}}{3}\,a^3$	$2\sqrt{3}\,a^2$	$\frac{\sqrt{2}}{2}\,a$	$\frac{\sqrt{6}}{6}\,a$
Dodeca-hedron	$\frac{15+7\sqrt{5}}{4}\,a^3$	$3\sqrt{5(5+2\sqrt{5})}\;a^2$	$\frac{(1+\sqrt{5})\sqrt{3}}{4}\,a$	$\frac{\sqrt{10+4.4\sqrt{5}}}{4}\,a$
Icosa-hedron	$\frac{5(3+\sqrt{5})}{12}\,a^3$	$5\sqrt{3}\,a^2$	$\frac{\sqrt{2(5+\sqrt{5})}}{4}\,a$	$\frac{(3+\sqrt{5})\sqrt{3}}{12}\,a$

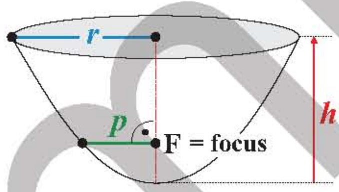
4.6 Solids with Curved Surface

Ellipsoid



► $V = \frac{4}{3}\pi \cdot a \cdot b \cdot c$

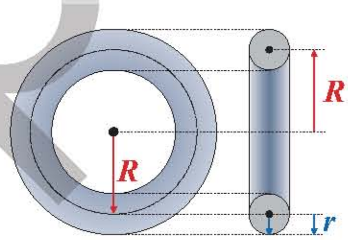
Paraboloid



► $V = \frac{1}{2}\pi \cdot r^2 \cdot h$

$= \pi \cdot p \cdot h^2$

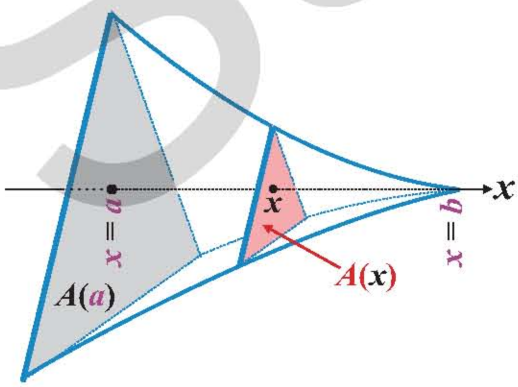
Torus



► $V = 2\,\pi^2 \cdot r^2 \cdot R$

► $A = 4\,\pi^2 \cdot r \cdot R$

4.7 Volume of a Solid using Integral Calculus



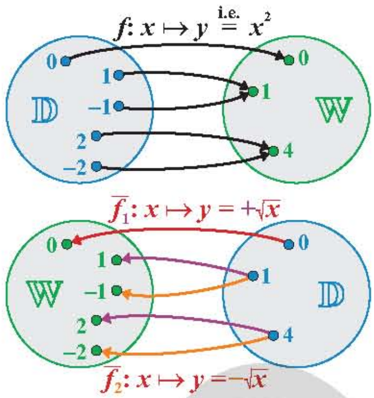
- $V = \int_a^b A(x)\,dx$

  cross-section area $A(x)$ ⊥ to the x-axis.

- Particularly solids of revolution: Volume of a solid obtained by the graph of a function $f(x)$ rotating about the x-axis:

  $V_x = \pi \cdot \int_a^b (\,f(x)\,)^2\,dx$ (see p. 25)

5 Functions



Definition: A **function** $f: \mathbb{D} \to \mathbb{W}$ is a **mapping** from one set $\mathbb{D}$ (**domain**) to another set $\mathbb{W}$ (**range**) so that **each** element $x \in \mathbb{D}$ is assigned **a unique** element $y \in \mathbb{W}$:

$$f: x \mapsto y = f(x)$$

Inverse function: $\overline{f}: \mathbb{W} \to \mathbb{D}$ reverses the function f:

$$\overline{f}(\, f(x)\,) = x \quad \text{and} \quad f(\,\overline{f}(y)\,) = y$$

Only one-to-one mappings have inverse functions. In order to make a function f invertible, its domain has to be restricted such that f becomes monotonic.

Finding the inverse function:

- *Graphically:* Reflect the graph in the first angle bisector $y = x$.
- *Algebraically:* Solve $y = f(x)$ for x. Then, interchange x and y.

Domain: Set of all allowed x-values:

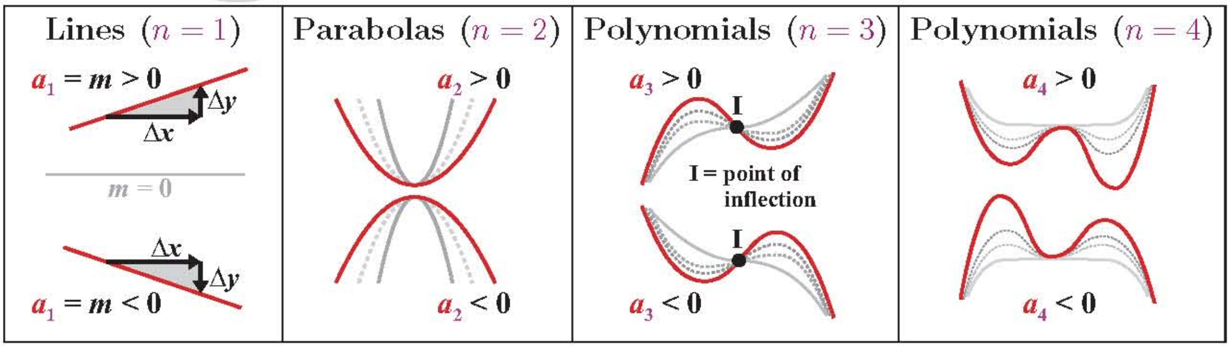
- $\dfrac{f(x)}{g(x)} \quad \Rightarrow \quad g(x) \neq 0$
- $\sqrt{g(x)} \quad \Rightarrow \quad g(x) \geq 0$
- $\log_a(g(x)) \quad \Rightarrow \quad g(x) > 0$

Table of functions and their inverse functions:

Function	$y = f(x)$	$\mathbb{D}_f$	$\mathbb{W}_f$	$y = \overline{f}(x)$
Reciprocal	$\frac{1}{x}$	$\mathbb{R}\setminus\{0\}$	$\mathbb{R}\setminus\{0\}$	$\frac{1}{x} = x^{-1}$
Square	x^2	$\mathbb{R}$	$x \geq 0$	$\sqrt{x} = x^{\frac{1}{2}}$
Power	x^n	$\mathbb{R}$	if n even: $x \geq 0$ if n odd: $\mathbb{R}$	$\sqrt[n]{x} = x^{\frac{1}{n}}$
Sine	$\sin(x)$	$\mathbb{R}$	$[-1,\ 1]$	$\arcsin(x)$
Cosine	$\cos(x)$	$\mathbb{R}$	$[-1,\ 1]$	$\arccos(x)$
Tangent	$\tan(x)$	$\mathbb{R}\setminus\{(n+\frac{1}{2})\,\pi,\ n \in \mathbb{Z}\}$	$\mathbb{R}$	$\arctan(x)$
Exponential	a^x	$\mathbb{R}$	$y > 0$	$\log_a(x)$

5.1 Polynomial Functions ("Parabolas of Degree n")

$$y = f(x) = a_n x^n + a_{n-1} x^{n-1} + \ \dots\ + a_1 x + a_0 = \sum_{k=0}^{n} a_k\, x^k$$ with $a_n \neq 0$, $\quad n$: degree (order).

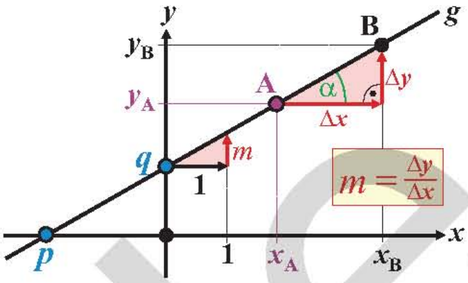


5.2 Linear Functions, Lines

▶ **Normal form:** $g:\ y = m \cdot x + q$

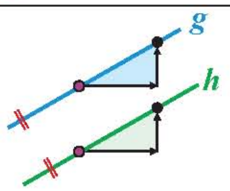
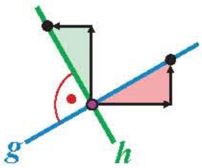
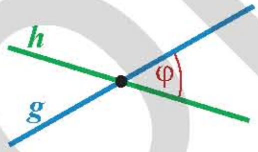
▶ **Point-slope form:** $g:\ y = m \cdot (x - x_A) + y_A$ with $A(x_A\ /\ y_A) \in g$.

- **Slope:** $m = \dfrac{\Delta y}{\Delta x} = \dfrac{y_B - y_A}{x_B - x_A} = \tan(\alpha)$
- ***y*-intercept:** q.



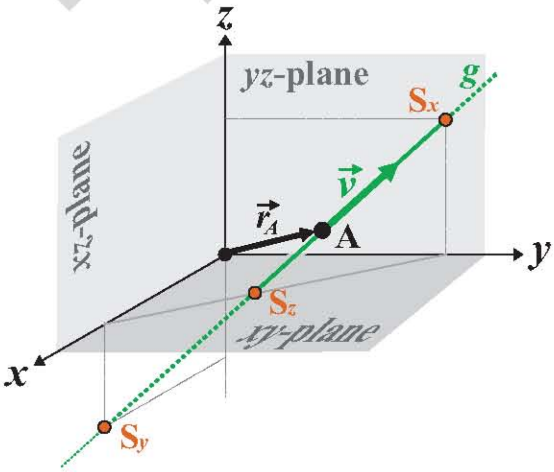
▶ **Intercept form:** $g:\ \dfrac{x}{p} + \dfrac{y}{q} = 1$ with

the axes intercepts $p,\ q\ \in\ \mathbb{R}\backslash\{0\}\ \cup\ \{\pm\infty\}$.

parallel lines	**perpendicular** lines	**angle of intersection** of g and h
		
$g \parallel h \ \Leftrightarrow m_g = m_h$	$g \perp h \ \Leftrightarrow m_g = \dfrac{-1}{m_g}$	$\tan(\varphi) = \left\lvert \dfrac{m_h - m_g}{1 + m_h \cdot m_g} \right\rvert$

▶ **Vector equation:** $g:\ t \mapsto \vec{r} = \vec{r}_A + t \cdot \vec{v}$

- **Direction vector** $\vec{v}$: arbitrary vector in the direction of g.
- **Support point:** arbitrary point A on g.
- **Track points** S_x, S_y, S_z: intersections of g with one of the main planes.



⇒ **Vector geometry see p. 27.**

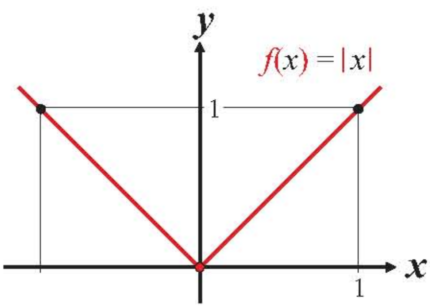
⇒ **Plane equations see p. 29.**

5.3 Absolute Value

$$|\,x\,| = \sqrt{x^2} = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

„makes x positive".

$|\,x\,|$ is continuous but not differentiable at $x = 0$.

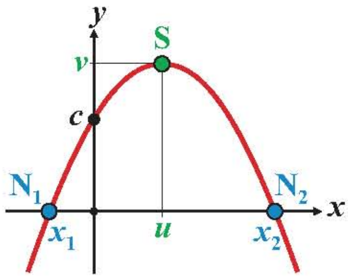


5.4 Quadratic Functions (Parabolas)

► **Normal form:**

$p: \; y = a\,x^2 + b\,x + c$

- $a < 0$: parabola opens downwards ($\cap$)
  $a > 0$: parabola opens upwards ($\cup$)
  $a = 1$: norm parabola
- $b\,x$: linear term
- c : y-intercept.



► **Vertex form:**

$p: \; y = a\,(x-u)^2 + v$

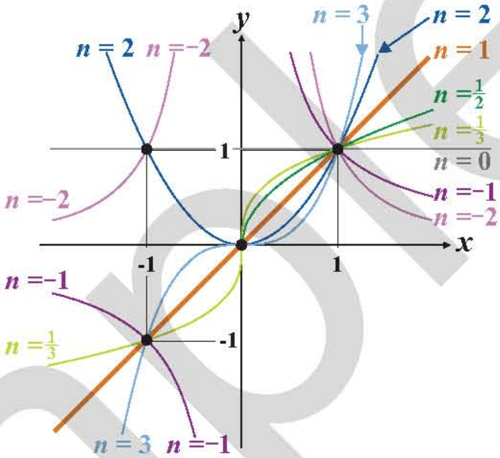
- a : see normal form.
- Vertex $S(u \,/\, v)$ with
  $u = -\frac{b}{2a}$; $\quad v = \frac{-b^2 + 4\,a\,c}{4\,a}$

$\Rightarrow$ Formula for quadratic equations see p. 19.

5.5 Power Functions

Power function: $f(x) = x^n$ $\quad n \in \mathbb{Q}$

$n = 0$	constant function.
$0 < n < 1$	root functions.
$n = 1$	linear function.
$n \in \mathbb{N};\; n > 1$	parabolas of n^{th} order.
$n \in \mathbb{N};\; n < 0$	hyperbolas of n^{th} order.



The graph of $f(x) = x^n$ is symmetrical about the $\begin{cases} y-\text{axis}, & \text{if } n \text{ is even.} \\ \text{origin}, & \text{if } n \text{ is odd.} \end{cases}$

$\Rightarrow$ Derivatives and antiderivatives see p. 24.

5.6 Rational Functions

A **rational function** $f(x)$ is a function of the following structure:

$$f(x) = \frac{U(x)}{V(x)} = \frac{\text{numerator polynomial}}{\text{denominator polynomial}} = \frac{a_n\,x^{\mathrm{n}} + a_{n-1}\,x^{n-1} + \ldots + a_1\,x + a_0}{b_m\,x^{\mathrm{m}} + b_{m-1}\,x^{m-1} + \ldots + b_1\,x + b_0}$$

coefficients $a_n, b_m \neq 0$.

$\mathbf{n} \in \mathbb{N}$: degree of numerator, $\quad \mathbf{m} \in \mathbb{N}\backslash\{0\}$: degree of denominator.

Properties:

► **vertical asymptotes (poles):** x_0 is called pole of f if

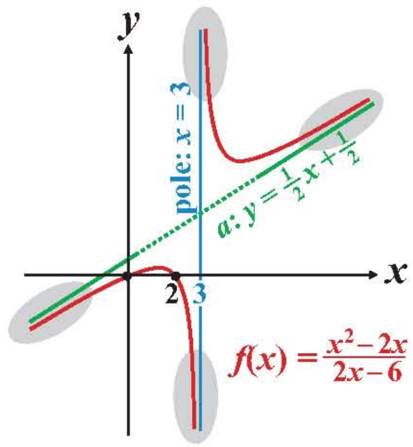
$y = \lim\limits_{x \to x_0} f(x) = \pm\infty$ (non- removable division by zero).

► **Horizontal or slant asymptote:** approaching function

$a(x)$ such that $\lim\limits_{x \to \pm\infty} \left(f(x) - a(x) \right) = 0$

For $n = m + 1$, $\quad a(x)$ is a slant linear asymptote.

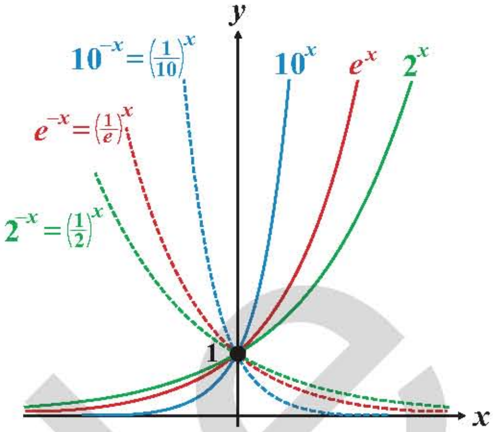
$\Rightarrow$ Limits see p. 21.



Linear f.
Power f.
Rational f.

5.7 Exponential and Logarithmic Functions

▶ **Exponential functions:** $y = f(x) = a^x$ $\quad a > 0.$



- **Euler's number:** $e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n \approx 2.718...$.
- **Groth / decay processes:** $N(t) = N_0 \cdot a^t$ with:

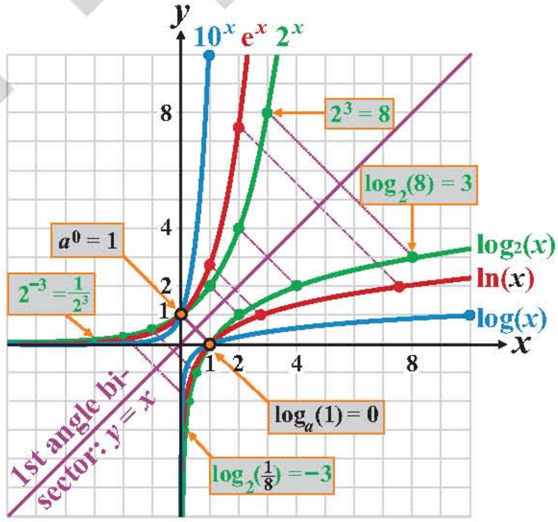
t	:	time
N_0	:	initial population at $t = 0$
$N(t)$	:	population at time t
a	:	groth factor: $a = 1 + \frac{p}{100}$ with
		p : groth in % per time unit.

⇒ See p. 5 for power and logarithm laws.
⇒ See p. 24 for derivatives and antiderivatives.

▶ **Logarithmic functions** (**logarithm laws see p. 5**):

$\overline{f}(x) = \log_a(x)$ $\quad x > 0$; $\quad a > 0; \quad a \neq 1.$

$\overline{f}(x) = \log_a(x)$ is the inverse function of $f(x) = a^x$:



- Common logarithm:

  $\overline{f}(x) = \log_{10}(x) = \log(x)$

  $\log(10^x) = x, \qquad 10^{\log(x)} = x \qquad (x > 0)$

- Natural logarithm:

  $\overline{f}(x) = \log_e(x) = \ln(x)$

  $\ln(e^x) = x, \qquad e^{\ln(x)} = x \qquad (x > 0)$

- Binary logarithm:

  $\overline{f}(x) = \log_2(x) = \text{lb}(x)$

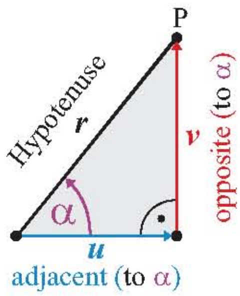
  $\log_2(2^x) = x, \qquad 2^{\log_2(x)} = x \qquad (x > 0)$

⇒ See p. 5 for power and logarithm laws.
⇒ See p. 24 for derivatives and antiderivatives.

5.8 Trigonometric Functions

▶ **Definition:** (see p. 6)

Right-angled triangle: $0 < \alpha < 90°$.

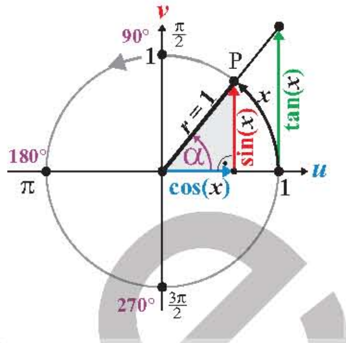


$$\sin(\alpha) = \frac{v}{r} = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos(\alpha) = \frac{u}{r} = \frac{\text{adjacent}}{\text{hypotenuse}}$$

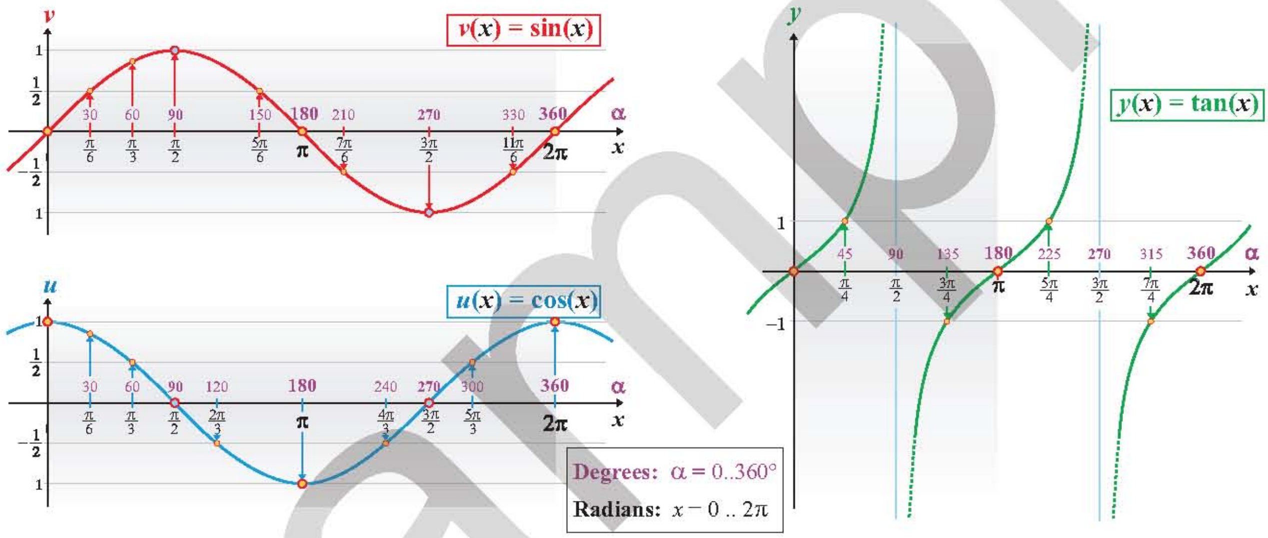
$$\tan(\alpha) = \frac{v}{u} = \frac{\text{opposite}}{\text{adjacent}} = \frac{\sin(\alpha)}{\cos(\alpha)}$$

Unit circle: $\alpha \in \mathbb{R}$.



Radians: $x = \alpha \cdot \frac{\pi}{180°}$ { Length of the arc in the unit circle corresponding to the central angle α.

▶ **Graphs:**



▶ **Properties and particular values:**

	$0° \doteq 0$	$30° \doteq \frac{\pi}{6}$	$45° \doteq \frac{\pi}{4}$	$60° \doteq \frac{\pi}{3}$	$90° \doteq \frac{\pi}{2}$	Periodicity	Symmetry
$\sin(x)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$360° \doteq 2\pi$ $\sin(x+2\pi n) = \sin(x)$	$\sin(\pi - x) = \sin(x)$ $\sin(-x) = -\sin(x)$
$\cos(x)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$360° \doteq 2\pi$ $\cos(x+2\pi n) = \cos(x)$	$\cos(2\pi - x) = \cos(x)$ $\cos(-x) = \cos(x)$
$\tan(x)$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	$(\pm\infty)$	$180° \doteq \pi$ $\tan(x+\pi n) = \tan(x)$	$\tan(-x) = -\tan(x)$

▶ **Domain:** $\mathbb{D}_{\sin} = \mathbb{D}_{\cos} = \mathbb{R}$ $\qquad \mathbb{D}_{\tan} = \mathbb{R}\backslash\{(\frac{\pi}{2} + n\pi),\ n \in \mathbb{Z}\}$.

▶ **Range:** $\mathbb{W}_{\sin} = \mathbb{W}_{\cos} = [-1, 1]$ $\quad \mathbb{W}_{\tan} = \mathbb{R}$.

▶ **Inverse functions:**
- $\arcsin(x)$ sometimes also $\sin^{-1}(x)$
- $\arccos(x)$ sometimes also $\cos^{-1}(x)$
- $\arctan(x)$ sometimes also $\tan^{-1}(x)$.

Exp / log f.
Trigonom. f.

please remove this part

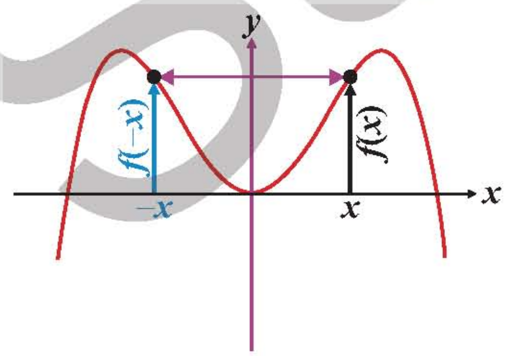
Identities and Properties of Trigonometric Functions:

$\tan(x) = \frac{\sin(x)}{\cos(x)}$	$\sin^2(x) + \cos^2(x) = 1$	$\frac{1}{\cos^2(x)} = 1 + \tan^2(x)$
$\sin(-x) = -\sin(x)$	$\cos(-x) = \cos(x)$	$\tan(-x) = -\tan(x)$
$\sin(\pi - x) = \sin(x)$	$\cos(\pi - x) = -\cos(x)$	$\tan(\pi - x) = -\tan(x)$
$\sin(\frac{\pi}{2} \pm x) = \cos(x)$	$\cos(\frac{\pi}{2} \pm x) = \mp\sin(x)$	$\tan(\frac{\pi}{2} \pm x) = \mp\frac{1}{\tan(x)}$
$\sin(2x) = 2\sin(x)\cos(x)$	$\cos(2x) = \begin{cases} 2\cos^2(x) - 1 \\ \cos^2(x) - \sin^2(x) \\ 1 - 2\sin^2(x) \end{cases}$	$\tan(2x) = \frac{2\tan(x)}{1 - \tan^2(x)}$
$\sin^2(\frac{x}{2}) = \frac{1 - \cos(x)}{2}$	$\cos^2(\frac{x}{2}) = \frac{1 + \cos(x)}{2}$	$\tan^2(\frac{x}{2}) = \frac{1 - \cos(x)}{1 + \cos(x)}$
$\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y)$	$\tan(x \pm y) = \frac{\tan(x) \pm \tan(y)}{1 \mp \tan(x) \cdot \tan(y)}$	
$\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y)$		
$\sin(x) + \sin(y) = 2\sin(\frac{x+y}{2})\cos(\frac{x-y}{2})$	$\sin(x) - \sin(y) = 2\cos(\frac{x+y}{2})\sin(\frac{x-y}{2})$	
$\cos(x) + \cos(y) = 2\cos(\frac{x+y}{2})\cos(\frac{x-y}{2})$	$\cos(x) - \cos(y) = -2\sin(\frac{x+y}{2})\sin(\frac{x-y}{2})$	

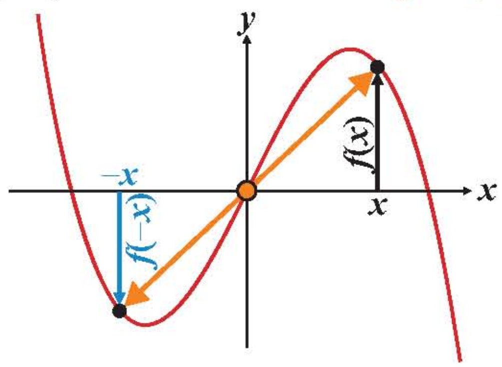
⇒ Derivatives and antiderivatives see p. 24.

5.9 Symmetry

Even functions:	**Odd functions:**
Graph symmetrical about the *y*-axis	Graph symmetrical about the origin O(0 / 0)



$f(-x) = f(x)$



$f(-x) = -f(x)$

6 Equations

6.1 Fundamental Theorem of Algebra

In $\mathbb{R}$, every polynomial of degree n can be written as a product of $k \leq n$ linear factors and irreducible quadratic factors $q(x) \neq 0$:

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0 \quad \Leftrightarrow \quad a_n \cdot (x - x_1) \cdot (x - x_2) \cdot \ldots \cdot (x - x_k) \cdot q(x) = 0$$

In the factorised form, the solutions (roots) x_1, $x_2, \ldots,$ x_k can be read.

6.2 Quadratic Equations

$ax^2 + bx + c = 0 \quad a, b, c \in \mathbb{R},\ a \neq 0$

Discriminant: $D = b^2 - 4\,a\,c$

Solutions: $x_{1,2} = \dfrac{-b \pm \sqrt{b^2 - 4\,a\,c}}{2\,a} \qquad D \geq 0$

Viète's formulas:

Product of solutions: $x_1 \cdot x_2 = \frac{c}{a}$

Sum of solutions: $x_1 + x_2 = -\frac{b}{a}$

6.3 Polynomial Equations of 3[rd] and Higher Degree

$ax^3 + bx^2 + cx + d = 0 \quad a, b, c, d \in \mathbb{R},\ a \neq 0.$

Method: Normalize to $a = 1$ (by division by $a \neq 0$), that is $x^3 + b'x^2 + c'x + d' = 0$. If there is an integer solution x_1, it must be a divisor of d'. Find solution x_1 by trying the divisors of d'. Then divide the equation by $(x - x_1)$ to find further solutions.

6.4 Numerical Methods to Calculate Zeros

To calculate a zero N(x_N / 0) of a function f, start with a guess x_1. Then, set up a recursive sequence x_1, x_2, x_3,... with limit x_N using one of the following methods:

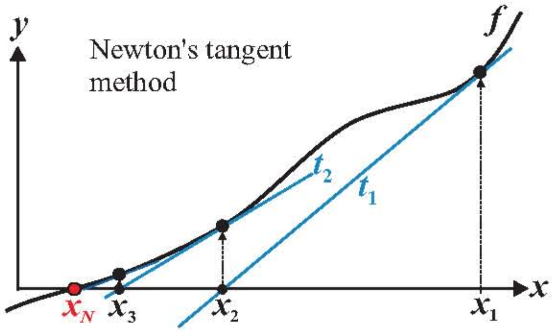
- **Secant method:** Choose $P_1(x_1 \ /\ f(x_1))$ and $P_2(x_2 \ /\ f(x_2))$ with $f(x_1) \cdot f(x_2) < 0$. Then:

$$x_{n+1} = x_n - f(x_1)\,\frac{x_2 - x_1}{f(x_2) - f(x_1)} \quad \xrightarrow[n\to\infty]{} \quad x_N$$

- **Newton's tangent method:** Choose $P_1(x_1 \ /\ f(x_1))$ with $f'(x_1) \neq 0$. Then:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \xrightarrow[n\to\infty]{} \quad x_N$$

The sequence is not always convergent.



Equations

7 Sequences and Series

Definition: A **sequence** is a **function** $a : \mathbb{N} \to \mathbb{R}$, $\quad n \mapsto a_n$. $\quad$ Notation:

- **Explicit definition:** $a_n = \{\text{formula in } n\}$.
- **Recursive definition:** $a_{n+1} = \{\text{formula in } a_n,\, a_{n-1}, \ldots\}$ and initial value a_1.

A **series** $s_1,\ s_2,\ s_3, \ldots$ is the sequence of partial sums of a given sequence $\{a_k\}_{k\in\mathbb{N}}$:

$$s_1 = a_1 \xrightarrow[+a_2]{} s_2 = a_1 + a_2 \xrightarrow[+a_3]{} s_3 = a_1 + a_2 + a_3 \quad \ldots \quad \boxed{s_n = \sum_{k=1}^{n} a_k}$$

7.1 Arithmetic Sequences and Series

	Recursive def.	Explicit definition
Sequence	$a_{n+1} = a_n + d$	$\boxed{a_n = a_1 + (n-1)\cdot d}$
Series	$s_{n+1} = s_n + a_{n+1}$	$\boxed{s_n = \frac{n}{2}\cdot(a_1 + a_n) = \frac{n}{2}\cdot(2a_1 + (n-1)\cdot d)}$

7.2 Geometric Sequences and Series

	Recursive def.	Explicit definition
Sequence	$a_{n+1} = a_n \cdot q$	$\boxed{a_n = a_1 \cdot q^{n-1}}$
Series	$s_{n+1} = s_n + a_{n+1}$	$\boxed{s_n = a_1 \cdot \frac{1-q^n}{1-q}}$ $\quad q \neq 1,\quad s_n = n\cdot a_1$ if $q = 1$ $\boxed{s = \lim\limits_{n\to\infty} s_n = \frac{a_1}{1-q}}$ $\quad$ if $\lvert q \rvert < 1$ $\quad$ (∞ geom. series)

7.3 Other Series

$$s_n = \sum_{k=1}^{n} \frac{1}{k^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2} = \frac{\pi^2}{6}$$

$$s_n = \sum_{k=1}^{n} \frac{1}{k} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \xrightarrow[n\to\infty]{} \infty \qquad \text{(Harmonic series)}$$

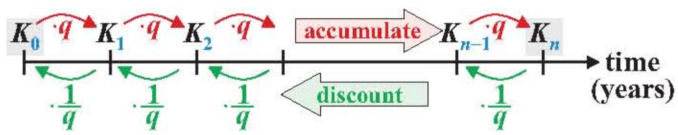
$$s_n = \sum_{k=1}^{n} k = \tfrac{1}{2}\,n\,(n+1) \qquad s_n = \sum_{k=1}^{n} k^2 = \tfrac{n}{6}\,(n+1)(2n+1) \qquad s_n = \sum_{k=1}^{n} k^3 = \left(\tfrac{1}{2}\,n\,(n+1)\right)^2$$

7.4 Capital with compound interest:

Seed capital K_0, duration n years:

Cash value:

$$\boxed{K_0 = K_n \cdot \frac{1}{q^n}}$$



Accumulated:

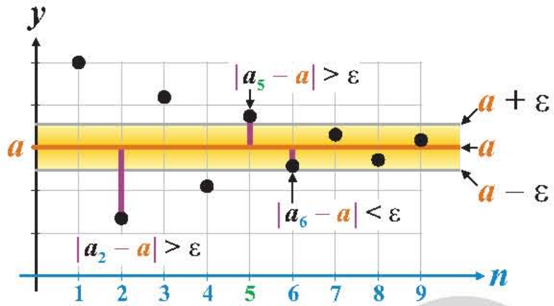
$$\boxed{K_n = K_0 \cdot q^n}$$

7.5 Limits

A sequence $\{a_n\}_{n\in\mathbb{N}}$ is called **convergent** with limit $a = \lim\limits_{n\to\infty} a_n$, if for **any** arbitrarily small number $\varepsilon > 0$ there is an index $N \in \mathbb{N}$, such that

$$|a_n - a| < \varepsilon$$

holds for all $n > N$. For arbitrarily large n, the distance between a_n and a tends to 0 (becomes smaller than any $\varepsilon > 0$).



- Sequences without limit (or such with $\lim\limits_{n\to\infty} a_n = \pm\infty$) are called **divergent**.
- **Undefined expressions:** $\frac{0}{0}$, $\frac{(\pm\infty)}{(\pm\infty)}$, $0\cdot(\pm\infty)$ and $\infty - \infty$

▶ **Limit identities:** Assuming that $a = \lim\limits_{n\to\infty} a_n$ and $b = \lim\limits_{n\to\infty} b_n$ exist:

- $\lim\limits_{n\to\infty} (a_n \pm b_n) = a \pm b$
- $\lim\limits_{n\to\infty} (c\cdot a_n) = c\cdot a$
- $\lim\limits_{n\to\infty} (a_n \cdot b_n) = a\cdot b$
- $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}$ if $b \neq 0$

⇒ Similar identities hold for limits $\lim\limits_{x\to x_0} f(x)$.

▶ **Limits of exponential functions:**

$$\lim_{x\to\infty} a^x = \begin{cases} 0, & \text{if } -1 < a < 1 \\ 1, & \text{if } a = 1 \\ \infty, & \text{if } a > 1 \end{cases}$$

▶ **Limits of rational functions:**

$$\lim_{x\to\infty} \frac{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \ldots + b_1 x + b_0} = \begin{cases} 0, & n < m \\ \frac{a_n}{b_m}, & n = m \\ \pm\infty, & n > m \end{cases}$$

▶ **Dominance rule:**

Exponential growth is faster than power growth: $\lim\limits_{x\to\infty} \frac{x^n}{e^x} = 0$

Power growth is faster than logarithmic growth: $\lim\limits_{x\to\infty} \frac{\ln(x)}{x^n} = 0$ for $n > 0$.

▶ **L'Hôpital's rule:** Assume $\lim\limits_{x\to x_0} f(x) = 0$ (or ∞) **and** $\lim\limits_{x\to x_0} g(x) = 0$ (or ∞), then:

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}$$

Example: $\lim\limits_{x\to 0} \frac{\sin(x)}{x} = \lim\limits_{x\to 0} \frac{\cos(x)}{1} = 1.$

7.6 Mean Values

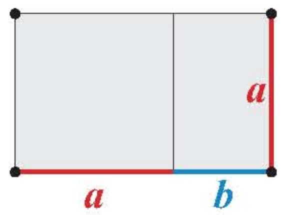
Let x_1, x_2 ,..., x_n be n given values.

- **Arithmetic mean value:** $\overline{x}_A = \frac{x_1 + x_2 + ... + x_n}{n}$ (see p. 33)
- **Weighted average value:** $\overline{x}_A = \frac{p_1 x_1 + p_2 x_2 + ... + p_n x_n}{p_1 + p_2 + ... + p_n}$

  where $p_1, p_2, \ldots, p_n$ are the relative frequencies of the values x_1, x_2 ,..., x_n.
- **Root mean square value:** $\overline{x}_{RMS} = \sqrt{\frac{x_1^2 + x_2^2 + ... + {x_n}^2}{n}}$
- **Geometric mean value:** $\overline{x}_G = \sqrt[n]{x_1 \cdot x_2 \cdot \ldots \cdot x_n}$
- **Harmonic mean value:** $\overline{x}_H = n \cdot \left(\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n} \right)^{-1}$ $x_k \neq 0.$
- Inequalities: $\overline{x}_H \leq \overline{x}_G \leq \overline{x}_A \leq \overline{x}_{RMS}$ hold, if $x_k \geq 0$ for all $k = 1, 2, \ldots n$.

7.7 Harmonic Section, Golden Ratio

Golden rectangle:



Two lines are in the Golden Ratio Φ if they intersect in the Harmonic Ratio: $\Phi = \frac{a}{b} = \frac{a+b}{a}$ therefore:

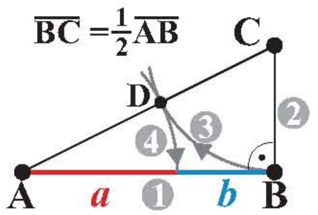
$\Phi^2 - \Phi - 1 = 0 \quad \Rightarrow \quad \Phi_{1,2} = \frac{1 \pm \sqrt{5}}{2} = \begin{cases} \Phi = 1.618... \\ \overline{\Phi} = -0.618... \end{cases}$

Properties:

- $\overline{\Phi} = -\frac{1}{\Phi}$
- Φ is irrational and can also be written as:

  $\Phi = \sqrt{1 + \sqrt{1 + \sqrt{1 + ...}}}$ $\qquad \Phi = 1 + \frac{1}{1 + \frac{1}{1 + ...}}$

Harmonic section of $\overline{AB}$:



7.8 Mathematical Induction

Method of mathematical proof for statements $\mathbb{A}_n$ on natural numbers ($\mathbb{N}$).

(I) **Base clause:** show that $\mathbb{A}_1$ is true. (Instead of $n = 1$, any other initial value for n may be taken. The proof holds beginning from the initial value.)

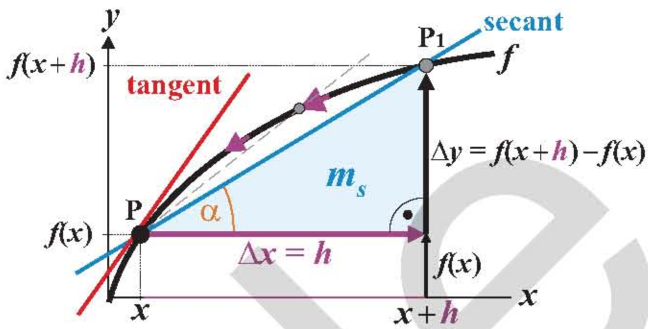
(II) **Recursive clause, step from n to $(n+1)$:**
calculate $\mathbb{A}_{n+1}$ recursively and show that the result coincides with the one calculated directly, that is $\mathbb{A}_{n+1}$ obtained by substituting n by $(n+1)$ in $\mathbb{A}_n$.

8 Differential Calculus

Assumption: Let $y = f(x)$ be a continuous function $f: \mathbb{R} \to \mathbb{R}, \quad x \mapsto y = f(x)$.

- **Slope of the secant, difference quotient:**
  average rate of change
  of f in the interval $[x,\ x + h]$:

$$m_s = \frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{h} = \tan(\alpha)$$



- **Slope of the tangent, differential quotient:**
  instantaneous rate of change
  gradient of f at $P(\ x\ /\ y = f(x)\ )$:
  Definition of the first derivative:

$$m_t = f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

8.1 Rules of Differentiation

Let $f(x)$, $u(x)$ and $v(x)$ be differentiable functions and c a constant.

► **Constant summand:** $f(x) = u(x) \pm c$

$$f'(x) = u'(x)$$

► **Product rule:** $f(x) = u(x) \cdot v(x)$

$$f'(x) = u'(x) \cdot v(x) + u(x) \cdot v'(x)$$

► **Constant factor:** $f(x) = c \cdot u(x)$

$$f'(x) = c \cdot u'(x)$$

► **Quotient rule:** $f(x) = \frac{u(x)}{v(x)}$

$$f'(x) = \frac{u'(x) \cdot v(x) - u(x) \cdot v'(x)}{(\ v(x)\ )^2}$$

► **Sum rule:** $f(x) = u(x) \pm v(x)$

$$f'(x) = u'(x) \pm v'(x)$$

► **Chain rule:** $f(x) = u(v(x))$

$$f'(x) = u'(v) \cdot v'(x) = \frac{du}{dv} \cdot \frac{dv}{dx}$$

"outer derivative times inner derivative".

8.2 Sufficient Criteria to Calculate Particular Points

		f	f'	f''	f'''
Zero	$N(x_N\ /\ 0)$	$f(x_N) = 0$	-	-	-
High point	$H(x_H\ /\ f(x_H))$		$f'(x_H) \overset{\bigstar}{=} 0$	$f''(x_H) \overset{\blacklozenge}{<} 0$	-
Low point	$L(x_L\ /\ f(x_L))$		$f'(x_L) \overset{\bigstar}{=} 0$	$f''(x_L) \overset{\blacklozenge}{>} 0$	-
Stationary point of inflection	$S(x_S\ /\ f(x_S))$		$f'(x_S) \overset{\bigstar}{=} 0$	$f''(x_S) \overset{\bigstar}{=} 0$	$f'''(x_S) \overset{\blacklozenge}{\neq} 0$
Inflection point	$I(x_I\ /\ f(x_I))$		-	$f''(x_I) \overset{\bigstar}{=} 0$	$f'''(x_I) \overset{\blacklozenge}{\neq} 0$

★ = necessary condition. (★ + ♦) = sufficient condition.

8.3 Table of Derivatives and Antiderivatives

antiderivative $F(x)$	function $f(x)$	1st derivative $f'(x)$
$\frac{x^{n+1}}{n+1}$ $[n \neq -1]$	x^n	$n \cdot x^{n-1}$
$\ln\lvert x \rvert$	$\frac{1}{x} = x^{-1}$	$-\frac{1}{x^2} = -x^{-2}$
$\frac{2}{3} \cdot x^{\frac{3}{2}}$	$\sqrt{x} = x^{\frac{1}{2}}$	$\frac{1}{2 \cdot \sqrt{x}}$
e^x	e^x	e^x
$x \cdot (\ln\lvert x \rvert - 1)$	$\ln\lvert x \rvert$	$\frac{1}{x} = x^{-1}$
$\frac{1}{\ln(a)} \cdot a^x$	a^x	$a^x \cdot \ln(a)$
$\frac{x}{\ln(a)} \cdot (\ln\lvert x \rvert - 1)$	$\log_a \lvert x \rvert$	$\frac{1}{x \cdot \ln(a)}$
Observe: Variable x in radians!		
$-\cos(x)$	$\sin(x)$	$\cos(x)$
$\sin(x)$	$\cos(x)$	$-\sin(x)$
$-\ln(\lvert \cos(x) \rvert)$	$\tan(x)$	$\frac{1}{\cos^2(x)} = 1 + \tan^2(x)$
$x \arcsin(x) + \sqrt{1 - x^2}$	$\arcsin(x)$	$\frac{1}{\sqrt{1-x^2}}$
$x \arccos(x) - \sqrt{1 - x^2}$	$\arccos(x)$	$-\frac{1}{\sqrt{1-x^2}}$
$x \arctan(x) - \frac{\ln(x^2+1)}{2}$	$\arctan(x)$	$\frac{1}{x^2+1}$

9 Integral Calculus

Let $F(x)$ be an **antiderivative (primitive)** of $f(x)$, that is a function satisfying $F'(x) = f(x)$. Then, any further antiderivative $F_1(x)$ of $f(x)$ may differ by an additive constant only: $F_1(x) = F(x) + C$. The constant C is called **constant of integration**.

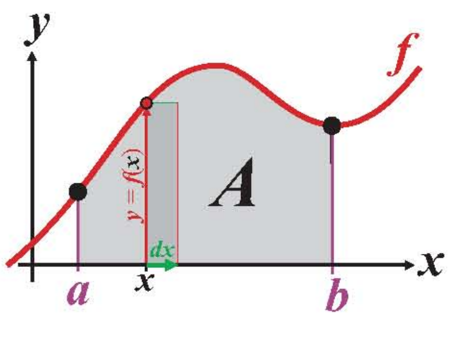
- **Indefinite integral: set of all antiderivatives:**

  $\int f(x)\,dx = \{F(x) + C \mid C \in \mathbb{R}\}$ with constant C.

- **Definite integral and Fundamental theorem of calculus:**

  $$A = \int_a^b f(x)\,dx = F(b) - F(a) = [F(x)]_a^b$$

  $\lvert A \rvert$: area between f and the x-axis between the integration limits $x = a$ and $x = b$, if f has no zero in $[a, b]$.



9.1 Rules of Integration

▶ **Constant rule:**
$$\int_a^b \left(c \cdot f(x)\right)\,dx = c \cdot \int_a^b f(x)\,dx$$

▶ **Sum rule:**
$$\int_a^b \left(u(x) \pm v(x)\right)dx = \int_a^b u(x)\,dx \pm \int_a^b v(x)\,dx$$

▶ **Orientation of integral:**
$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

▶ **Interval additivity:**
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

▶ **Integration by parts:**
$$\int_a^b u(x) \cdot v'(x)\,dx = \left[\,u(x) \cdot v(x)\,\right]_a^b - \int_a^b u'(x) \cdot v(x)\,dx$$

▶ **Substitution rule:** Let $f(x) = u(\,v(x)\,)$ be a composite function. $U(v)$ denotes an anti-derivative of the outer function $u(v)$. Then:
$$\int_a^b u(\,v(x)\,) \cdot v'(x)\,dx = \int_{v(a)}^{v(b)} u(v)\,dv = \left[\,U(v)\,\right]_{v(a)}^{v(b)}$$

9.2 Volume of a Solid of Revolution and Arc Length

- **Rotation about x-axis:** $V_x = \pi \int_a^b (\,f(x)\,)^2\,dx$

  generalization see p. 12.

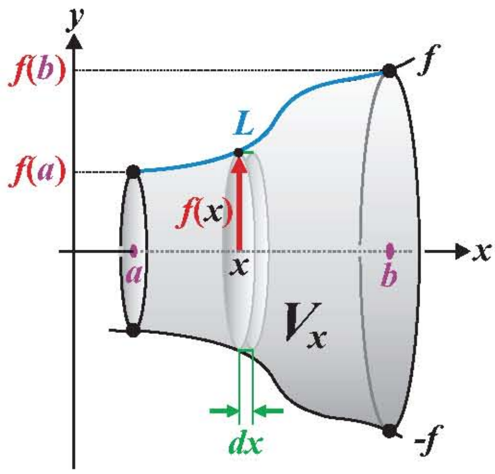
- **Rotation about y-axis:** $V_y = \pi \int_{f(a)}^{f(b)} (\,\overline{f}(y)\,)^2\,dy$

  $y = f(x)$ strictly monotone.

  $x = \overline{f}(y)$ is the inverse function of $y = f(x)$.

  ⇒ Inverse function see p. 13.

- **Arc length:** $L = \int_a^b \sqrt{1 + (\,f'(x)\,)^2}\,dx$



9.3 Power Series, Taylor Polynomials

► **Taylor polynomial** $T_n(x)$: approximation of a function $f(x)$ at x_0 by a polynomial

of n^{th} degree: $$T_n(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x_0)\,(x - x_0)^k$$

where $f^{(k)}(x)$ denotes the k^{th} derivative of f. In detail:

$$T_n(x) = f(x_0) + f'(x_0)\,(x - x_0) + \frac{1}{2!} f''(x_0)\,(x - x_0)^2 + \ldots + \frac{1}{n!} f^{(n)}(x_0)\,(x - x_0)^n$$

Remainder term: $$R_n(x) = f(x) - T_n(x) = \frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n+1)}(x_0 + \alpha(x - x_0)), \quad 0 < \alpha < 1.$$

► **Power Series:**

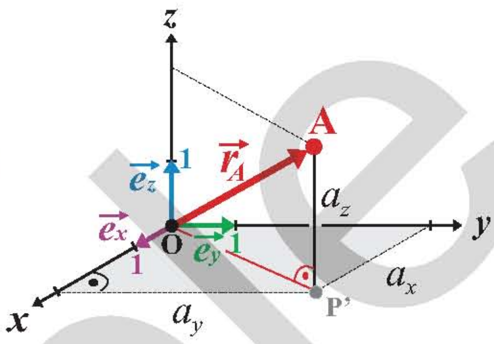
term	power series	valid for
$(1+x)^n$	$1 + \binom{n}{1} x + \binom{n}{2} x^2 + \binom{n}{3} x^3 + \ldots$	$n \in \mathbb{N};\quad \lvert x \rvert < 1$
$\frac{1}{1+x}$	$1 - x + x^2 - x^3 \pm \ldots$	$\lvert x \rvert < 1$
$\sqrt{1+x}$	$1 + \frac{1}{2} x - \frac{1}{2 \cdot 4} x^2 + \frac{1 \cdot 3}{2 \cdot 4 \cdot 6} x^3 - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8} x^4 \pm \ldots$	$\lvert x \rvert < 1$
e^x	$1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \frac{1}{4!} x^4 + \ldots$	$x \in \mathbb{R}$
$\ln(x)$	$(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 \mp \ldots$	$0 < x \le 2$
$\sin(x)$	$x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \frac{1}{7!} x^7 \pm \ldots$	$x \in \mathbb{R}$
$\cos(x)$	$1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 - \frac{1}{6!} x^6 \pm \ldots$	$x \in \mathbb{R}$
$\tan(x)$	$x + \frac{1}{3} x^3 + \frac{2}{15} x^5 + \frac{17}{315} x^7 + \ldots$	$\lvert x \rvert < \frac{\pi}{2}$
$\arcsin(x)$	$x + \frac{1}{2 \cdot 3} x^3 + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5} x^5 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 7} x^7 + \ldots$	$\lvert x \rvert \le 1$
$\arctan(x)$	$x - \frac{1}{3} x^3 + \frac{1}{5} x^5 - \frac{1}{7} x^7 \pm \ldots$	$\lvert x \rvert < 1$

10 Vector Geometry

Definition: A vector $\vec{r}_A$ describes a **translation or displacement** from O to A. Vectors have a length (magnitude, absolute value) and an orientation (direction). Vectors can be arbitrarily parallel shifted, that is vectors do *not* have a fixed initial point.

▶ **Standard unit vectors:**

$$\vec{e}_x = \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad \vec{e}_y = \begin{pmatrix}0\\1\\0\end{pmatrix}, \quad \vec{e}_z = \begin{pmatrix}0\\0\\1\end{pmatrix}$$



▶ **Linear combination:** Every 3-dimensional vector $\vec{r}_A$ can be written as a linear combination of $\vec{e}_x$, $\vec{e}_y$, $\vec{e}_z$:

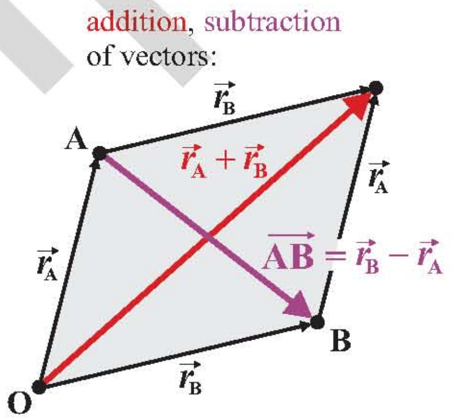
$$\vec{r}_A = \begin{pmatrix}a_x\\a_y\\a_z\end{pmatrix} = a_x \cdot \vec{e}_x + a_y \cdot \vec{e}_y + a_z \cdot \vec{e}_z.$$

a_x, a_y, a_z are called the components of $\vec{r}_A$.

▶ **Position vector** of $A(a_x, a_y, a_z)$: $\vec{r}_A = \overrightarrow{OA} = \begin{pmatrix}a_x\\a_y\\a_z\end{pmatrix}$: { vector from the origin to point A.

▶ **Magnitude, length:**

$$|\vec{r}_A| = r_A = \overline{OA} = \sqrt{a_x^2 + a_y^2 + a_z^2}$$



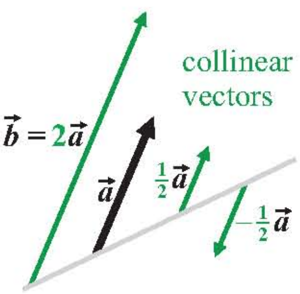
▶ **Addition, subtraction:**

$$\vec{r}_A \pm \vec{r}_B = \begin{pmatrix}a_x\\a_y\\a_z\end{pmatrix} \pm \begin{pmatrix}b_x\\b_y\\b_z\end{pmatrix} = \begin{pmatrix}a_x \pm b_x\\a_y \pm b_y\\a_z \pm b_z\end{pmatrix}$$

(Vector difference) = (position vector to final point) − (position vector to initial point)

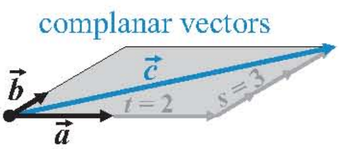
▶ **Multiplication by a scalar (number), collinear vectors $\vec{a}$ and $\vec{b}$:**

$$\vec{b} = k \cdot \vec{a} = k \cdot \begin{pmatrix}a_x\\a_y\\a_z\end{pmatrix} = \begin{pmatrix}k \cdot a_x\\k \cdot a_y\\k \cdot a_z\end{pmatrix}$$



Complanar vectors: $\vec{c}$ is complanar to $\vec{a}$ and $\vec{b}$ if $\vec{c}$ can be written as a linear combination of $\vec{a}$ and $\vec{b}$:

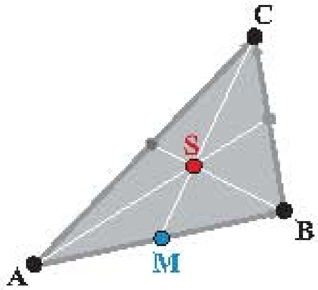
There are $t, s \in \mathbb{R}$ such that $\vec{c} = t \cdot \vec{a} + s \cdot \vec{b}$ holds.



► **Midpoint of A and B:** $\vec{r}_M = \frac{1}{2}\,(\vec{r}_A + \vec{r}_B)$

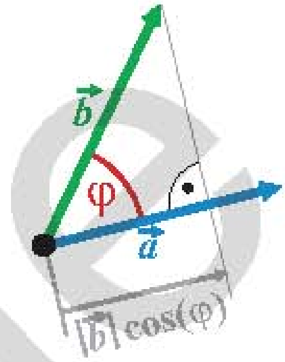
► **Centroid of $\triangle$ABC:** $\vec{r}_S = \frac{1}{3}\,(\vec{r}_A + \vec{r}_B + \vec{r}_C)$

center of mass, see also p. 7.



► **Scalar product (dot product):**
(perpendicular projection of $\vec{b}$ to $\vec{a}$)

$$\vec{a}\cdot\vec{b} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} \cdot \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} = a_x b_x + a_y b_y + a_z b_z = |\vec{a}|\cdot|\vec{b}|\cdot\cos(\varphi)$$

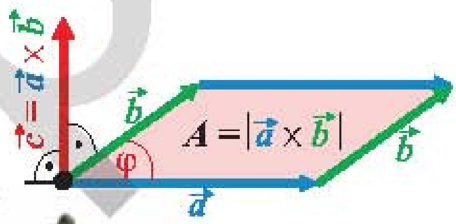


► **Angle φ between $\vec{a}$ and $\vec{b}$:** $\cos(\varphi) = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|}$

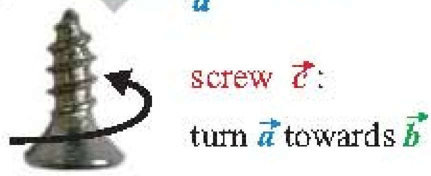
► **Perpendicular vectors:** $\vec{a}\perp\vec{b} \Leftrightarrow \vec{a}\cdot\vec{b} = 0$ if $\vec{a}, \vec{b} \neq \vec{0}$.

► **Vector product (cross product):**

$$\vec{c} = \vec{a}\times\vec{b} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} \times \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} = \begin{pmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{pmatrix}$$



$\vec{c}\perp\vec{a}$ and $\vec{c}\perp\vec{b}$



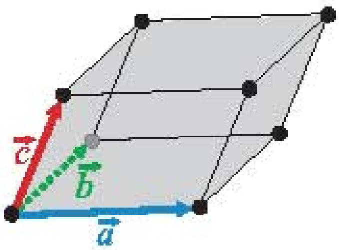
$$|\vec{c}| = |\vec{a}\times\vec{b}| = |\vec{a}|\cdot|\vec{b}|\cdot\sin(\varphi)$$

$|\vec{c}|$: area of the parallelogram defined by $\vec{a}$ and $\vec{b}$.

► **Triple product:**

$$V = |(\vec{a}\times\vec{b})\cdot\vec{c}| = |(\vec{b}\times\vec{c})\cdot\vec{a}| = |(\vec{c}\times\vec{a})\cdot\vec{b}|$$

V : volume of the parallelepiped
defined by $\vec{a}$, $\vec{b}$ and $\vec{c}$.



⇒ **Linear functions, lines see p. 14.**

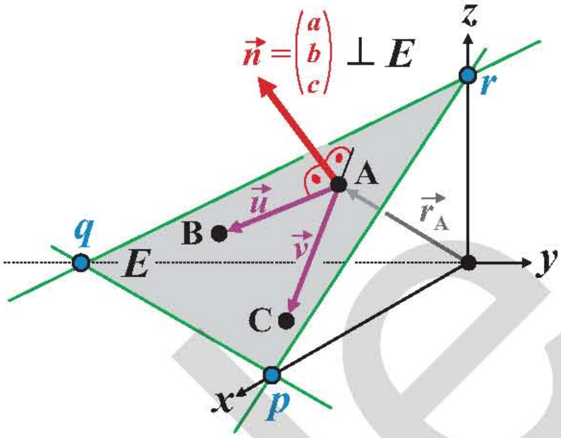
⇒ **Plane equations see p. 29.**

10.1 Planes

▶ **Vector equation:** $E: \vec{r} = \vec{r}_A + t \cdot \vec{u} + s \cdot \vec{v}$

- If **3 points** A, B, C or a **support point** A (position vector $\vec{r}_A$) and **two independent directions** $\vec{u} = \overrightarrow{AB}$ and $\vec{v} = \overrightarrow{AC}$ are known.
- Each pair $t, s \in \mathbb{R}$ corresponds to exactly one point P (position vector $\vec{r}$) on E.
- **Normal vector:**

$$\vec{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \vec{u} \times \vec{v} \perp E$$ (see p. 28).



▶ **Normal form:** $E: \vec{n} \cdot (\vec{r} - \vec{r}_A) = 0$

▶ **Cartesian form:** $E: a \cdot x + b \cdot y + c \cdot z + d = 0$

- **Hesse's normal form:** $H(x, y, z) = \dfrac{a \cdot x + b \cdot y + c \cdot z + d}{\sqrt{a^2 + b^2 + c^2}} = 0$
- **Distance** P$(u \,/\, v \,/\, w)$ **to** E: $d(\mathrm{P}, E) = \dfrac{|a \cdot u + b \cdot v + c \cdot w + d|}{\sqrt{a^2 + b^2 + c^2}}$

▶ **Intercept form:** $E: \frac{x}{p} + \frac{y}{q} + \frac{z}{r} = 1$ with the intercepts $p, q, r \in \mathbb{R}\backslash\{0\} \cup \{\pm\infty\}$.

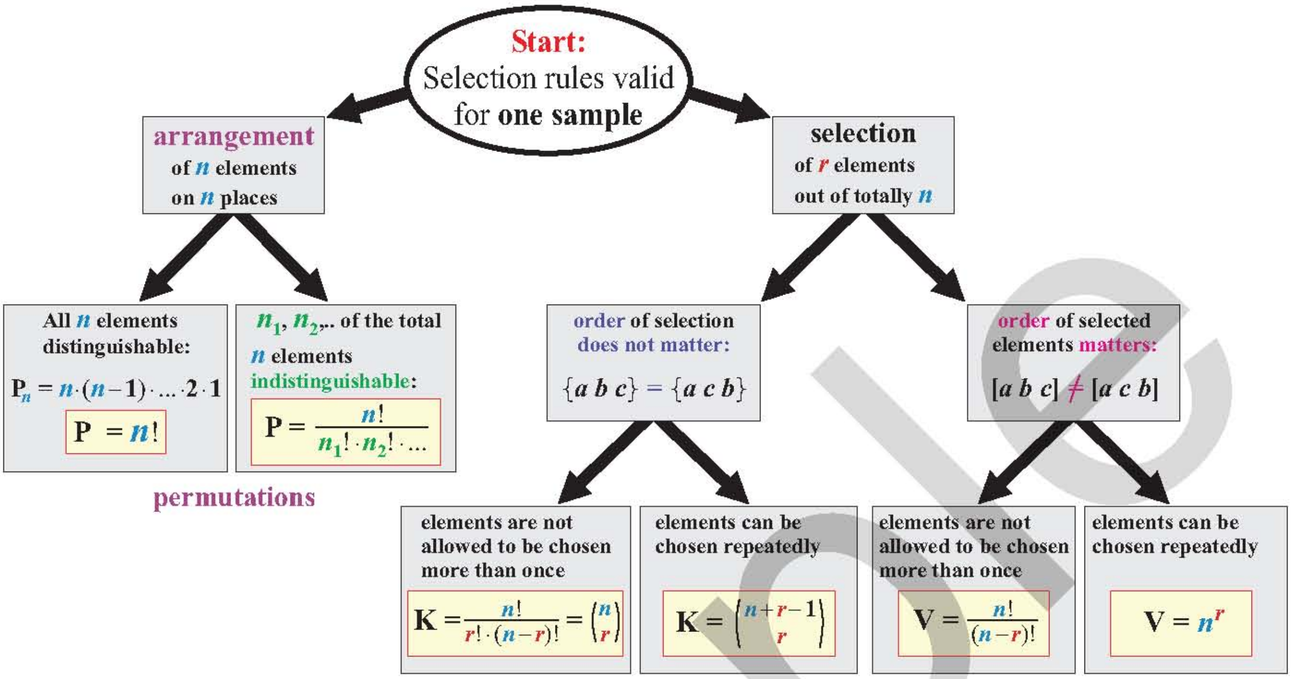
Plane E parallel to the x-axis: $E: \frac{y}{q} + \frac{z}{r} = 1$ with cyclic permutations:

⇒ **Linear functions, lines see p. 14.**

⇒ **Vector geometry see p. 27.**

11 Stochastics

11.1 Combinatorics



Factorial: $n! = 1 \cdot 2 \cdot \ldots \cdot n$ $\quad 0! = 1$, $1! = 1$

Binomial coefficient: $\binom{n}{r} = \frac{n!}{r! \cdot (n-r)!}$

Symmetry: $\binom{n}{r} = \binom{n}{n-r}$

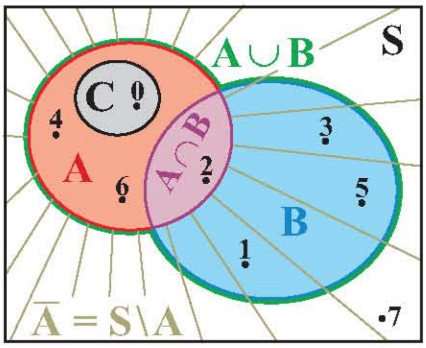
Recurrence relation: $\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$

11.2 Probability and Set Theory

- **Sample space S:** set of all possible outcomes.
- **Events A, B, C:** subsets of S.

  Ex.: $S = \{0, 1, 2, 3, 4, 5, 6, 7\}$, $A = \{0, 2, 4, 6\}$, $B = \{1, 2, 3, 5\}$, $\quad 4 \in A$; $3 \notin A$.

$\lvert A \rvert$	cardinality	number of elements in A
$A \cap B$	intersection	A and B
$A \cup B$	union	A or B
$\overline{A} = G \setminus A$	complement	G without A
$A \setminus B$	difference	A without B
$C \subset A$	subset	C contained in A
$\{\}$, $\emptyset$	empty set	



- **Laplace-probability:** If all elements in S have the same probability to occur, then:

$$p(A) = \frac{|A|}{|S|} = \frac{\text{number of elements in A}}{\text{number of elements in S}} = \frac{\text{favorable}}{\text{possible}}$$

Impossible event $p(\emptyset) = 0$ **Certain event** $p(S) = 1$	$0 \leq p(A) \leq 1$
Complementary probability	$p(\overline{A}) = 1 - p(A)$ $\Rightarrow$ Venn diagram see p. 30.
Addittion law	$p(A \cup B) = p(A) + p(B) - p(A \cap B)$
Conditional probability	$p(B \mid A)$: probability that B occurs under the condition that A has already occurred: A = IF, B = THEN: $p(B \mid A) = \frac{\lvert A \cap B \rvert}{\lvert A \rvert} = \frac{p(A \cap B)}{p(A)}$ (reduction of the sample space from S to A)
Multiplication law	$p(A \cap B) = p(A) \cdot p(B \mid A)$
Independent events	Two events A and B are **independent** if $p(A \cap B) = p(A) \cdot p(B)$ holds.

$\Rightarrow$ Binomial distribution see p. 32.

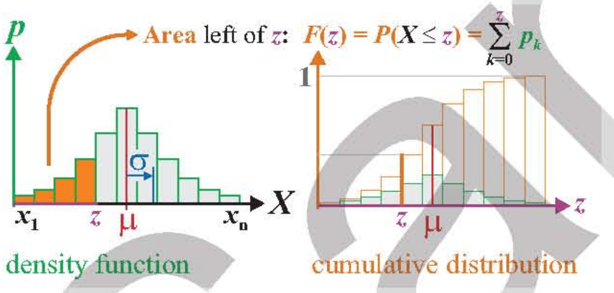
11.3 Probability Distributions

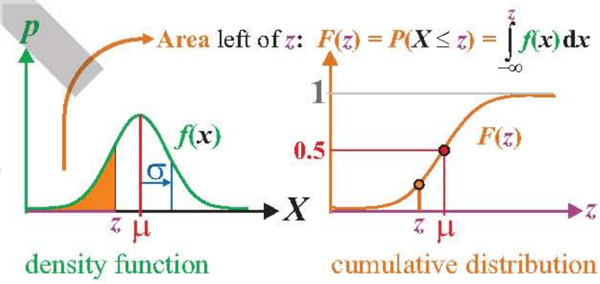
Discrete random variable:

The **random variable** X takes only n discrete values $x_1, x_2, \ldots, x_n$ with the probabilities $p_1, p_2, \ldots, p_n$.

Continuous random variable:

The **random variable** X may take all values $x \in \mathbb{R}$. The density function $f(x)$ evaluates the probability for exactly x. *Notice:* strictly speaking, this probability is always zero.





Discrete		Continuous
$p_1 + p_2 + \ldots + p_n = \sum_{k=1}^{n} p_k = 1$	**Normalization**	$\int_{-\infty}^{\infty} f(x)\,dx = 1$
$\mu = E(X) = \sum_{k=1}^{n} p_k\, x_k$	**Expected value (mean value)**	$\mu = E(X) = \int_{-\infty}^{\infty} f(x) \cdot x\,dx$
$\sigma^2 = \sum_{k=1}^{n} p_k\,(x_k - \mu)^2$	**Variance**	$\sigma^2 = \int_{-\infty}^{\infty} f(x) \cdot (x - \mu)^2\,dx$
$\sigma = \sqrt{\sigma^2} = \sqrt{\mathrm{var}(X)}$	**Standard deviation**	$\sigma = \sqrt{\sigma^2} = \sqrt{\mathrm{var}(X)}$

Let X, Y be two random variables and a, b constants. Then:

$E(a \cdot X + b \cdot Y) = a \cdot E(X) + b \cdot E(Y)$

$\mathrm{var}(a \cdot X + b) = a^2 \cdot \mathrm{var}(X)$

11.4 Binomial Distribution (Discrete Distribution)

The sample space of an experiment, which is repeated n times, consists of exactly two elements: $S = \{A, \overline{A}\}$ with constant probabilities $p(A) = p$ and $p(\overline{A}) = 1 - p$. Let X be the number of times A occurs in totally n repetitions. Then:

A occurs **at least once**	$P(X \geq 1) = 1 - (1-p)^n$	
A occurs **exactly *r* times**	$P(X = r) = \binom{n}{r} \cdot p^r \cdot (1-p)^{n-r}$	$0 \leq r \leq n$
A occurs **at most *x* times**	$P(X \leq x) = \sum_{r=0}^{x} \binom{n}{r} \cdot p^r \cdot (1-p)^{n-r}$	$0 \leq x \leq n$
Mean value	$E(X) = n \cdot p$	
Standard deviation	$\sigma = \sqrt{n \cdot p \cdot (1-p)}$	

For $\sigma^2 = n \cdot p \cdot (1-p) > 9$ the binomial distribution can be approximated by a normal distribution.

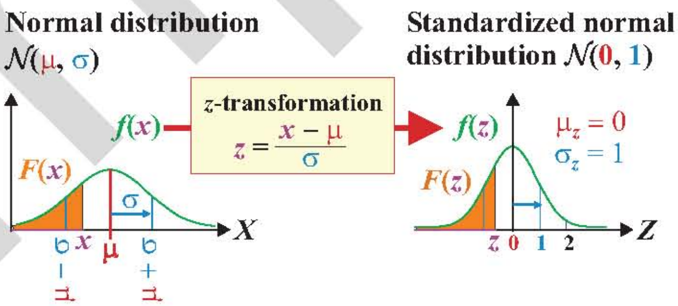
11.5 Normal Distribution (Continuous Distribution)

- **Density function:**

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}} = N(\mu, \sigma)$$

Standardized normal distribution:

$$f(z) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{z^2}{2}} = N(0, 1)$$



Symmetry:

$f(\mu + x) = f(\mu - x)$ $\quad$ $f(-z) = f(+z)$

$F(-z) = 1 - F(+z)$

- **Cumulative normal distribution:**

$$F(x) = P(X \leq x) = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{x} e^{-\frac{(t-\mu)^2}{2\sigma^2}}\, dt$$

probability of observing *at most* x.

Standardized normal distribution:

$$F(z) = P(Z \leq z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{t^2}{2}}\, dt$$

⇒ see the table in the inner cover.

- **σ-environments for the normal distribution:**

$1\,\sigma$-environment	$2\,\sigma$-environment	$3\,\sigma$-environment
$P(\lvert X-\mu \rvert < 1\,\sigma) \approx 68.3\%$	$P(\lvert X-\mu \rvert < 2\,\sigma) \approx 95.4\%$	$P(\lvert X-\mu \rvert < 3\,\sigma) \approx 99.7\%$

12 Statistics

12.1 Univariate Data (one Variable)

$X = \{x_1,\ x_2,\ \ldots,\ x_k\}$ denotes the values of a sample and $n_1,\ n_2,\ \ldots,\ n_k$ their **absolute frequency** of size $n = \sum\limits_{i=1}^{k} n_i = n_1 + n_2 + \ldots + n_k$. The **relative frequencies** $p(x_i) = \frac{n_i}{n}$ behave like the Laplace-probability of observing the value x_i, particularly $\sum\limits_{i=1}^{k} p(x_i) = 1$.

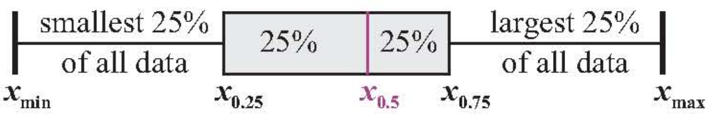
	Individual data	**Grouped data (classes)**
Data	n values $x_1,\ x_2,\ \ldots,\ x_n$	k values $x_1,\ x_2,\ \ldots,\ x_k$ with abs. frequency $n_1,\ n_2,\ \ldots,\ n_k$
Arithmetic mean (expected value)	$\overline{x} = E(X) = \frac{1}{n}\sum\limits_{i=1}^{n} x_i$	$\overline{x} = E(X) = \frac{1}{n}\sum\limits_{i=1}^{k} n_i\, x_i = \sum\limits_{i=1}^{k} p(x_i)\cdot x_i$
Median	The median $x_{0.5}$ of the values of an ordered sample is • the value of the middle item, if n is odd. • the mean of the middle two items, if n is even.	
Mode	The mode x_M is the most frequent value observed in a sample.	
Range	$R = x_{\max} - x_{\min}$	
Variance [*]	$s_x^2 = \frac{1}{n-1}\sum\limits_{i=1}^{n}(x_i - \overline{x})^2$	$s_x^2 = \frac{1}{n-1}\sum\limits_{i=1}^{k} n_i\,(x_i - \overline{x})^2$ or $s_x^2 = \sum\limits_{i=1}^{k} p(x_i)\,(x_i - \overline{x})^2 = E(X^2) - \left(E(X)\right)^2$

[*] If the values $x_1,\ x_2,\ \ldots,\ x_n$ represent an entire population or if we are interested in the variation within the sample itself, the denominator is n (instead of $n-1$).

Standard deviation: $s_x = \sqrt{s_x^2}$

The **variation coefficient** $V = \frac{s_x}{\overline{x}} \cdot 100\%$ is used to compare different samples.

Box plot: Evaluate the median $x_{0.5}$, the upper ($x_{0.75}$) and lower ($x_{0.25}$) quartiles, the smallest ($x_{\min}$) and the largest ($x_{\max}$) sample. Then



Inequality of Chebychev:

For a sample with mean $\overline{x}$ and variance s_x^2, the probability p for an observation x to be found within a range of $\pm\lambda$ from the mean is given by $p(|\,x - \overline{x}\,| < \lambda)\ \geq\ 1 - \frac{s_x^2}{\lambda^2}$

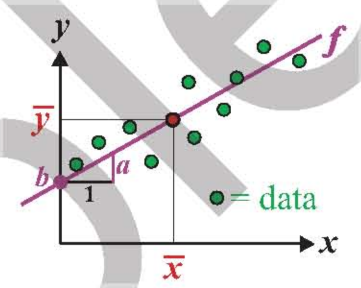
12.2 Bivariate Data (two Variables): Regression and Correlation

Let (x_1, y_1), (x_2, y_2), $\ldots (x_n, y_n)$ be n pairs of observations. To describe the dependency between x and y a model function $y = f(x)$ which depends on the parameters a, b,... is fitted to the data such that the mean square deviation of $y_i - f(x)$ becomes a minimum:

$$F(X, Y, a, b, \ldots) = \sum_{i=1}^{n} \left(y_i - f(x_i)\right)^2 \quad \longrightarrow \quad \text{minimum}$$

Linear Regression:

Model function: $y = f(x) = ax + b$ with



- **slope** $$a = \frac{\sum_{i=1}^{n}(x_i - \overline{x})\,(y_i - \overline{y})}{\sum_{i=1}^{n}(x_i - \overline{x})^2} = \frac{c_{xy}}{s_x^2} = r_{xy} \cdot \frac{s_y}{s_x}$$
- **$y-$intercept** $$b = \overline{y} - a \cdot \overline{x}$$

Correlation coefficient:

$$r_{xy} = \frac{\sum_{i=1}^{n}(x_i - \overline{x})\,(y_i - \overline{y})}{\sqrt{\sum_{i=1}^{n}(x_i - \overline{x})^2 \cdot \sum_{i=1}^{n}(y_i - \overline{y})^2}} = \frac{c_{xy}}{s_x \cdot s_y}$$

Covariance:

$$c_{xy} = \frac{1}{n-1} \sum_{i=1}^{n}(x_i - \overline{x})\,(y_i - \overline{y})$$

$$c_{xy} = E(x \cdot y) - E(x) \cdot E(y)$$

Alternative: System of linear equations to calculate a and b of the linear model function:

$$\left| \begin{array}{lcl} \left(\sum_{i=1}^{n} x_i^2\right) \cdot a + \left(\sum_{i=1}^{n} x_i\right) \cdot b & = & \sum_{i=1}^{n} x_i \cdot y_i \\ \left(\sum_{i=1}^{n} x_i\right) \cdot a + \quad n \quad \cdot b & = & \sum_{i=1}^{n} y_i \end{array} \right|$$

Index